BAMO Bay Area Mathematical Olympiad

18th Bay Area Mathematical Olympiad

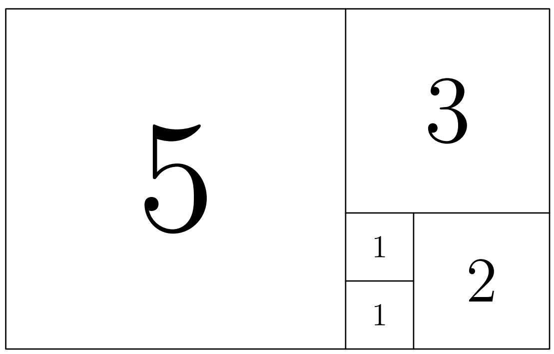
February 23, 2016

Problems and Solutions

BAMO-8 and BAMO-12 are each 5-question essay-proof exams, for middle- and high-school students, respectively. The problems in each exam are in roughly increasing order of difficulty, labeled A through E in BAMO-8 and 1 through 5 in BAMO-12, and the two exams overlap with three problems. Hence problem C on BAMO-8 is problem #1 on BAMO-12, problem D on BAMO-8 is #2 in BAMO-12, and problem E in BAMO-8 is #3 in BAMO-12.

The solutions below are sometimes just sketches. There are many other alternative solutions. We invite you to think about alternatives and generalizations!

A The diagram below is an example of a *rectangle tiled by squares*:



Each square has been labeled with its side length. The squares fill the rectangle without overlapping.

In a similar way, a rectangle can be tiled by nine squares whose side lengths are 2, 5, 7, 9, 16, 25, 28, 33, and 36. Sketch a possible arrangement of those squares. They must fill the rectangle without overlapping. Label each square in your sketch by its side length, as in the picture above.

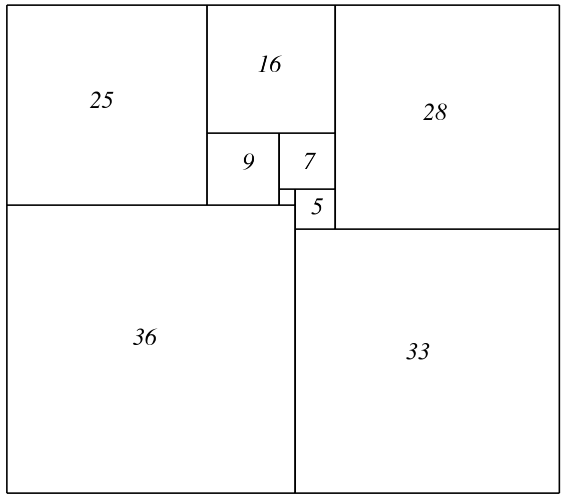
Solution: To tile a rectangle, the areas of the squares must add to match the area of the rectangle. The total area of the 9 squares is:

$$2^2+5^2+7^2+9^2+16^2+25^2+28^2+33^2+36^2=4209.$$

When we factor 4209 we obtain $4209=3\times 23\times 61$.

To fit the largest square, the rectangle has to be at least 36 units wide and high, and the only way to do that with these three factors is a rectangle of size 61×69.

It's easy to get started by noticing that $33+36=69$ and with so few squares the two largest squares must lie on one edge of the rectangle of length 69. From there it becomes easy to place the other ones, and the following figure illustrates a possible solution (the unlabeled square has side length of 2). There are actually 3 additional solutions that are obtained from this one by either a rotation of 180° or taking a mirror image across either the horizontal or vertical axis.



B A weird calculator has a numerical display and only two buttons, $\boxed{D\sharp}$ and $\boxed{D\flat}$. The first button doubles the displayed number and then adds 1. The second button doubles the displayed number and then subtracts 1. For example, if the display is showing 5, then pressing the sequence $\boxed{D\sharp}\boxed{D\flat}\boxed{D\sharp}\boxed{D\sharp}$ will result in a display of 87.

(a) Suppose the initial displayed number is 1. Give a sequence of exactly eight button presses that will result in a display of 313.

(b) Suppose the initial displayed number is 1, and we then perform exactly eight button presses. What are all the numbers that can possibly result? Prove your answer by showing that all of these numbers can be produced, and that no other numbers can be produced.

Solution:

(a) The sequence (unique) is $\boxed{D\sharp}\boxed{D\flat}\boxed{D\flat}\boxed{D\sharp}\boxed{D\sharp}\boxed{D\sharp}\boxed{D\flat}\boxed{D\flat}$, producing intermediate values $3, 5, 9, 19, 39, 79, 157, 313$.

(b) The numbers produced will be the odd numbers from 1 to 511, inclusive. Clearly the only numbers producible are odd, and certainly the minimum and maximum values will be 1 and 511, respectively, since that is what is produced after 7 presses of $\boxed{D\flat}$ and 7 presses of $\boxed{D\sharp}$, respectively. We still need to prove that *any* odd number between 1 and 511 can produced.

There are many possible methods. Here is one. Observe that if two numbers are different and we press the same button, we have to get two different outputs. Likewise, if we press different buttons with the same starting value, we get different outputs. However, it is possible to start with two different numbers a, b and get the same output, as long as we apply

$\boxed{\text{D}\flat}$ to the larger one and $\boxed{\text{D}\sharp}$ to the smaller one. For example, suppose that $b > a$. Then we could have $2b - 1 = 2a + 1$, which is equivalent to $b = a + 1$. However, this will never happen, since the numbers that we have to work with are always odd!

In other words, we start with 1, and after one button press, we have two possible outputs, namely 1 and 3. Applying $\boxed{\text{D}\flat}$ to these two will yield two distinct odd results, and applying $\boxed{\text{D}\sharp}$ to them will also yield two distinct odd results, and there can be *no overlap*! So after two button presses, we will have 4 distinct odd numbers, starting with 1 and ending with 7. Clearly this pattern persists to 7 presses. At each stage, we get twice as many distinct odd numbers, and after 7 presses, we will have $2^8 = 256$ different odd numbers, namely the set from 1 to 511. ■

C/1 The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5.

Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k > 1$).

Show that, for every choice of k,

(a) A and B have the same set of distinct prime factors.

(b) $A+1$ and $B+1$ have the same set of distinct prime factors.

Solution:

(a) Since B is given as a multiple of A, every prime that divides A also divides B.

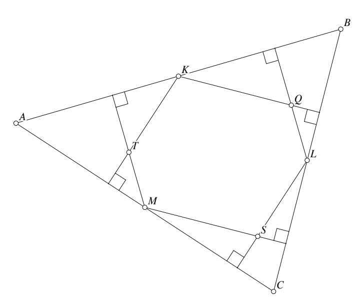
Conversely, suppose p is a prime that divides B. Since $B = 2^k \cdot A$, either p divides 2^k or p divides A. If p divides 2^k, then $p = 2$. But then p divides A anyway, because $A = 2(2^{k-1} - 1)$. This shows that every prime that divides B also divides A.

Since every prime that divides A also divides B, and vice versa, A and B have the same set of distinct prime factors.

(b) Observe that $A + 1 = 2^k - 1$ and $B + 1 = 2^k(2^k - 2) + 1 = 2^{2k} - 2 \cdot 2^k + 1 = (2^k - 1)^2 = (A+1)^2$. Since $B + 1 = (A+1)^2$, every prime that divides $A+1$ divides $B+1$ and vice versa.

Therefore, $A+1$ and $B+1$ have the same set of distinct prime factors. ■

D/2 In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc. The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon $KQLSMT$ inside triangle ABC. Prove that the area of the hexagon $KQLSMT$ is half of the area of the original triangle ABC.

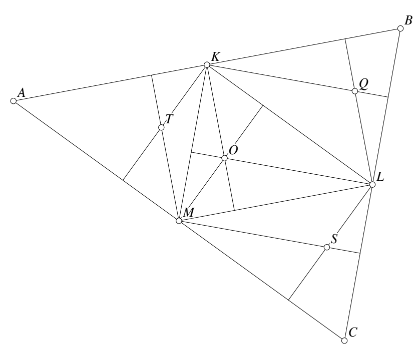


Solution: Construct segments KL, LM and MK. Next, construct the three altitudes of triangle KLM that meet in its orthocenter, O. Since KL connects the two midpoints of AB and BC we know that $KL \parallel AB$ it is easy to see that $MT \parallel KO \parallel LQ$ since those lines are perpendicular to a pair of parallel lines. Similarly, $KT \parallel MO \parallel LS$ and $KQ \parallel LO \parallel MS$.

Note that triangle KLM divides the original triangle ABC into four congruent parts, all similar to the original triangle (and thus all acute), so the perpendiculars we dropped from K, L, and M are just altitudes of the smaller triangles. Since all the smaller triangels are acute, all the altitudes will meet inside the respective triangles.

Because of all the parallel lines, we know that $OLSM$, $OMTK$ and $OKQL$ are all parallelograms having diagonals ML, KM, and LK, respectively. The diagonal of a parallelogram divides it into two congruent triangles, so triangle LSM is congruent to triangle MOL, triangle MTK is congruent to triangle KOM, and triangle KQL is congruent to triangle LOK. If we consider the hexagon $KQLSMT$ to be composed of triangle KLM and the three outer pieces, we can see that triangle KLM is composed of three smaller triangles that are congruent to the corresponding outer pieces, so the area of the hexagon is twice the area of triangle KLM.

But triangle KLM connects the midpoints of the edges of triangle ABC so each of its sides is half the length of the corresponding side of triangle ABC, so triangle KLM is similar to triangle ABC, but with $1/4$ the area. We previously showed that the area of $KQLSMT$ is twice the area of triangle KLM, so the area of the hexagon is $1/2$ the area of triangle ABC. ■



E/3 For $n > 1$, consider an $n \times n$ chessboard and place pieces at the centers of different squares. (a) With $2n$ chess pieces on the board, show that there are 4 pieces among them that form the vertices of a parallelogram.

(b) Show that there is a way to place $(2n-1)$ chess pieces so that no 4 of them form the vertices of a parallelogram.

Solution: (a) Since there can be at most n pieces that are leftmost in their rows (some rows may be empty), there are at least n pieces that are not the leftmost in their row. Record the distances (the number of squares) between the leftmost piece and the other pieces on the same row.

There are then at least n distances recorded. Since the distances range from 1 to $n-1$, by the Pigeonhole Principle, at least two of these distances are the same. This implies that there are at least two rows each containing two pieces that are the same distance apart. These four pieces yield a parallelogram. ■

(b) If $(2n-1)$ pieces are placed, for example, on the squares of the first column and the first row, then there is no parallelogram.

4 Find a positive integer N and $a_1, a_2, \ldots, a_N$, where $a_k = 1$ or $a_k = -1$ for each $k = 1, 2, \ldots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible.

Solution: It is possible, as long as the sum S desired is a multiple of 48, with $N = S/6$, which in this case is 3360336, and the a_k repeats the 8-term pattern $-1, 1, 1, -1, 1, -1, -1, 1$.

Use the observation that if $f(x)$ is a degree-k polynomial, then for any constant h, the difference $f(x+h) - f(x)$ will be a degree-$(k-1)$ polynomial. If we iterate this process three times, we can find a way to manipulate consecutive cubes to always get a *constant*.

More precisely, let

$$c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7 \ldots$$

be consecutive cubes. In other words, $c_m = (m+u)^3$ for some fixed starting integer u. Then the differences

$$c_1 - c_0, \quad c_3 - c_2, \quad c_5 - c_4, \quad c_7 - c_6, \ldots$$

will be quadratic functions; i.e., if we define $a_m := c_{m+1} - c_m$, then a_m is a quadratic function of m (depending on the parameter u, as well), and the differences are $a_0, a_2, a_4, a_6, \ldots$. Continuing, we see that the differences

$$a_2 - a_0, \quad a_6 - a_4, \ldots$$

will be a linear sequence; i.e., $b_m := a_{m+2} - a_m$ is a linear function of m (with parameter u), and our differences are $b_0, b_4, \ldots$. Finally, the sequence

$$b_4 - b_0, \quad b_{12} - b_8, \ldots$$

is constant, no matter what the parameter u equals! We have

$$b_4 - b_0 = a_6 - a_4 - (a_2 - a_0) = a_6 - a_4 - a_2 + a_0 = c_7 - c_6 - c_5 + c_4 - c_3 + c_2 + c_1 - c_0,$$

and since this is constant, we can compute it using any value of u. Taking $u = -3$, the constant must equal

$$4^3 - 3^3 - 2^2 + 1^3 - 0^3 + (-1)^3 + (-2)^3 - (-3)^3 = 48.$$

In other words, if we define

$$s_u := -u^3 + (u+1)^3 + (u+2)^3 - (u+3)^3 + (u+4)^3 - (u+5)^3 - (u+6)^3 + (u+7)^3,$$

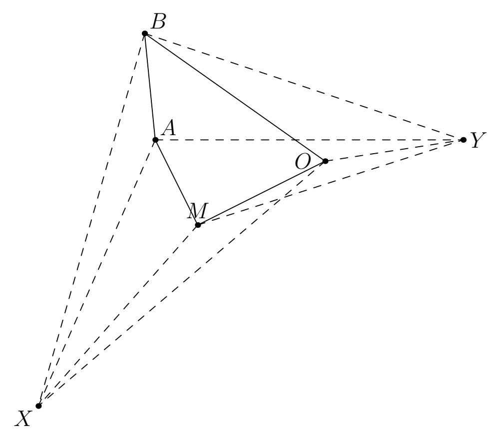
then $s_u = 48$ for all values of u.

Since $20162016 = 420042 \cdot 48$, we can easily write 20162016 as a sum of 420042 8-element sum/differences of $420042 \cdot 8 = 3360336$ consecutive cubes:

$$\begin{aligned}20162016 = \sum_{k=0}^{420041} s_{8k+1} &= -1^3+2^3+3^3-4^3+5^3-6^3-7^3+8^3\\ &\quad -9^3+10^3+11^3-12^3+13^3-14^3-14^3+15^3\\ &\quad \ldots\\ &\quad -3360329^3+3360330^3+\cdots-3360335^3+3360336^3. \quad \blacksquare\end{aligned}$$

NOTE: About half of the correct solutions were purely computational, making use of the fact that $1^3+2^3+\ldots+n^3=(n(n+1))^2/4$ and putting $n=95$ gets you to a number that is rather close to 20162016. It is then possible to work out—with great difficulty, by hand—values that work. Clearly this method is not generalizable, unlike the multiple-of-48 method above.

5 The corners of a fixed convex (but not necessarily regular) n-gon are labeled with distinct letters. If an observer stands at a point in the plane of the polygon, but outside the polygon, they see the letters in some order from left to right, and they spell a "word" (that is, a string of letters; it doesn't need to be a word in any language). For example, in the diagram below (where $n=4$), an observer at point X would read "$BAMO$," while an observer at point Y would read "$MOAB$."



Determine, as a formula in terms of n, the maximum number of distinct n-letter words which may be read in this manner from a single n-gon. Do not count words in which some letter is missing because it is directly behind another letter from the viewer's position.

Solution: Let us call our original n points $V_1,V_2,\ldots,V_n$.

If A,B are two points, then viewers on one side of line $\overleftrightarrow{AB}$ see A to the left of B, and viewers on the other side see B to the left of A. Therefore, if we draw the $\binom{n}{2}$ lines determined by pairs $\{V_i,V_j\}$ $(1\le i<j\le n)$, then different "views" of $V_1,V_2,\ldots,V_n$ (from outside their convex hull) are in one-to-one correspondence with the regions formed outside the convex hull by these $\binom{n}{2}$ lines. These regions are what we will now count. Our strategy is to count all regions, then subtract those regions that are inside the convex hull of $V_1,V_2,\ldots,V_n$.

We begin by stating and proving a general lemma:

Lemma. Let K be a convex region of the plane. Let r lines pass through the interior of K, making a total of m intersections in the interior of K. Suppose further that no three of the lines meet at one of these intersections. Then K is divided by the lines into $1+r+m$ regions.

Proof of Lemma. Number the lines $\ell_1, \ell_2, \ldots, \ell_r$. We imagine that we draw the lines one at a time, in this order. Suppose m_i is the number of points in the interior of K where ℓ_i intersects $\ell_1, \ell_2, \ldots, \ell_{i-1}$. Then at the stage when we draw ℓ_i, it passes through $m_i + 1$ existing regions, dividing each of them into two regions and thereby increasing the number of regions by $m_i + 1$. Since there is initially one region (K itself), the final number of regions after all r lines are drawn is

$$1 + (m_1 + 1) + (m_2 + 1) + (m_3 + 1) + \cdots + (m_r + 1).$$

But observe that $m_1 + m_2 + \cdots + m_r = m$, since each intersection point in the interior of K is counted exactly once on each side of the equation. Thus the final number of regions is $1 + r(1) + (m_1 + m_2 + \cdots + m_r) = 1 + r + m$. $\square$

By the lemma, if $\binom{n}{2}$ lines are in general position (no two parallel, no three concurrent), then they divide the plane into $1 + \binom{n}{2} + \binom{\binom{n}{2}}{2}$ regions.

However, our $\binom{n}{2}$ lines aren't in general position; $n-1$ lines meet at each of our original points V_n. Each V_i is surrounded by $2(n-1)$ regions, but if we nudged each line by a tiny amount so as to separate all their pairwise intersections, then these $2(n-1)$ regions would become $1 + n + \binom{n-1}{2}$ regions (by the lemma). Accounting for this, the number of regions formed by our $\binom{n}{2}$ lines is $1 + \binom{n}{2} + \binom{\binom{n}{2}}{2} + n\left[2(n-1) - \left(1 + n + \binom{n-1}{2}\right)\right]$.

Finally, we subtract the regions inside the convex hull of $V_1, V_2, \ldots, V_n$. The number of lines passing through the convex hull (i.e. diagonals, not sides) is $\binom{n}{2} - n$. Every set of four points $\{V_i, V_j, V_k, V_m\}$ ($1 \le i < j < k < m \le n$) determines a unique intersection inside the convex hull, so there are $\binom{n}{4}$ such intersections. Thus by the lemma, the convex hull is cut into $1 + \binom{n}{2} - n + \binom{n}{4}$ regions. Subtracting this from our total count of regions in the plane, we conclude that the number of regions outside the convex hull (which is our final answer) is

$$1 + \binom{n}{2} + \binom{\binom{n}{2}}{2} + n\left[2(n-1) - \left(1 + n + \binom{n-1}{2}\right)\right] - \left[1 + \binom{n}{2} - n + \binom{n}{4}\right].$$

This may be simplified to $\boxed{\frac{1}{12}n(n-1)(n^2 - 5n + 18)}$. ■

ALTERNATIVELY, there is another expression that this is equal to, namely

$$2\binom{n}{2} + 2\binom{n}{4}.$$

This more elegant expression was derived by two students, but only one gave an equally elegant combinatorial solution that made it easy to see why this counts the number of regions. For this work, Swapnil Garg won the brilliancy award for 2016. We leave it to the reader to discover his argument.