3rd Bay Area Mathematical Olympiad

February 27, 2001

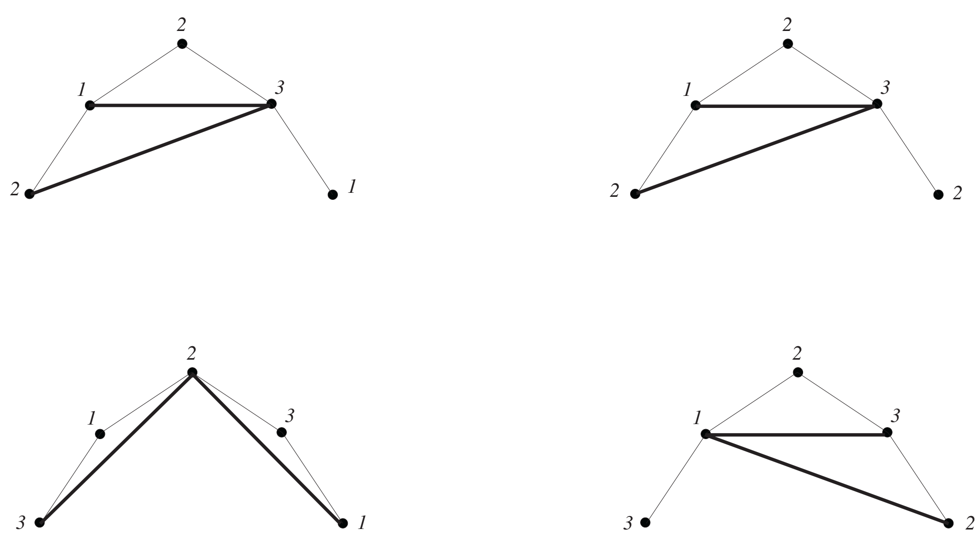
Problems and Solutions

1 Each vertex of a regular 17-gon is colored red, blue, or green in such a way that no two adjacent vertices have the same color. Call a triangle "multicolored" if its vertices are colored red, blue, and green, in some order. Prove that the 17-gon can be cut along nonintersecting diagonals to form at least two multicolored triangles.

(A *diagonal* of a polygon is a a line segment connecting two nonadjacent vertices. Diagonals are called *nonintersecting* if each pair of them either intersect in a vertex or do not intersect at all.)

Solution: (One of many similar solutions.) Denote the colors by 1, 2, 3. Notice that all three colors must be used. This is true because if only two colors were used, the vertex coloring would be of the type $121212\cdots$ which is impossible, since 17 is odd (this would make two adjacent vertices the same color). Hence there are three consecutive vertices colored (without loss of generality) 123, respectively. (For otherwise, if 3 consective vertices had coloration of the form *aba*, the next three vertices [overlapping two vertices] would have to be colored *bab*, etc., forcing the pattern *ababab* . . . which is not possible with an odd number of vertices.

Thus we have 4 cases depending on the colors of the vertices adjacent to this segment: 21231, 21232, 31231, 31232. It is easy to see that the desired construction can be done in each case; the figure below illustrates this.



■

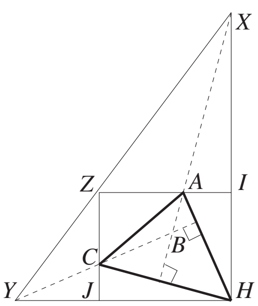
REMARK: You may wish to try a more general problem: show that there exists a triangulation of the 17-gon for which all triangles are multicolored. (A triangulation is a dissection of an entire polygon into triangles, where each triangle is formed from vertices and diagonals (or sides) of the original polygon, and no triangles overlap [besides sharing sides or vertices]. There are many ways to triangulate a given n-gon, but all of them use $(n-2)$ triangles.)

2 Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y and Z are collinear (lie on the same line).

Solution: Observe that $\angle XAI = \angle XHC = \angle HCJ$. Hence $\triangle XAI \sim \triangle HCJ$, and thus $XI/HJ = AI/CJ$. Likewise, $YJ/HI = CJ/AI$. Putting these together yields $XI/HJ = HI/YJ$, and hence

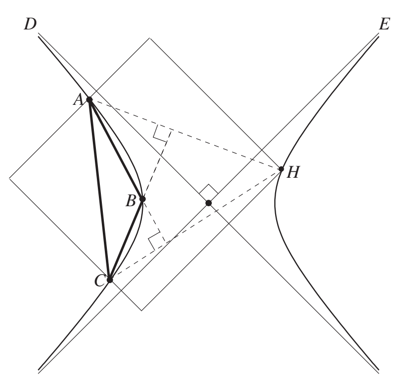
$$\frac{XI}{ZI} = \frac{ZJ}{YJ} \;\Rightarrow\; \triangle XZI \sim \triangle ZYJ.$$

Since $\angle JZI = 90^\circ$, this immediately implies $\angle YZX = 180^\circ$, and X, Y, Z are collinear.



■

REMARK: It turns out that this problem is equivalent to the **Poncelet–Brianchon Theorem**: *Let A, B and C be three points on a rectangular hyperbola (a hyperbola with perpendicular asymptotes.) Then the orthocenter of $\triangle ABC$ also lies on the hyperbola.*



Proof: In the projective plane, let D and E be the two points of intersection of the line at infinity l_∞ with the two asymptotes of the hyperbola Λ. Note that since Λ is a conic, D and E are also the points of intersection of Λ with l_∞. We apply :

Converse of Pascal's theorem: *If the three pairs of opposite sides in a hexagon intersect in collinear points, then the hexagon is inscribed in a conic.*

Note that through any 5 points in general position (no 3 of them collinear), there passes a unique conic. In our situation, the points A, B, C, D and E lie on a hyperbola, (and hence they are in general position). To show that the orthocenter H of $\triangle ABC$ also lies on this hyperbola, it suffices to verify that the points X, Y and Z are collinear, where $X = AB \cap HD$, $Y = BC \cap HE$, and $Z = AE \cap CD$. Thus, the problem is equivalent to

Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Then X, Y and Z are collinear.

But this was our problem #2! ∎

3 Let $f(n)$ be a function satisfying the following three conditions for all positive integers n:

(a) $f(n)$ is a positive integer,

(b) $f(n+1) > f(n)$,

(c) $f(f(n)) = 3n$.

Find $f(2001)$.

We will present two solutions. The first one was presented by the problem proposer, and was the method used essentially by all the students but one. The second solution, which won the BAMO 2001 Brilliancy Prize, is due to Andrew Dudzik of Lynbrook HS. Andrew's solution, (which we only sketch) is not shorter, nor does it use any ingenious mathematical tricks. But it is superior to the other solutions, because it illuminates the nature of $f(n)$ in a very simple way.

Solution 1: We will show that $f(2001)$ must equal 3816. We start by proving a lemma which gives us some of the values of $f(n)$.

Lemma: *For $n = 0, 1, 2, \ldots$,*

(a) $f(3^n) = 2 \cdot 3^n$*; and*

(b) $f(2 \cdot 3^n) = 3^{n+1}$.

Proof: We use induction. For $n = 0$, note that $f(1) \neq 1$, otherwise $3 = f(f(1)) = f(1) = 1$, which is impossible. Since $f(k)$ is a positive integer for all positive integers k, we conclude that $f(1) > 1$. Since $f(n+1) > f(n)$, f is increasing. Thus $1 < f(1) < f(f(1)) = 3$ or $f(1) = 2$. Hence $f(2) = f(f(1)) = 3$.

Suppose that for some positive integer $n \geq 1$,

$$f\left(3^n\right) = 2 \cdot 3^n \text{ and } f\left(2 \cdot 3^n\right) = 3^{n+1}.$$

Then,

$$f(3^{n+1}) = f\left(f(2 \cdot 3^n)\right) = 2 \cdot 3^{n+1},$$

and

$$f(2 \cdot 3^{n+1}) = f\left(f(3^{n+1})\right) = 3^{n+2},$$

as desired. This completes the induction, and establishes the lemma.

Continuing with our solution, there are $3^n - 1$ integers m such that $3^n < m < 2 \cdot 3^n$ and there are $3^n - 1$ integers m' such that

$$f\left(3^n\right) = 2 \cdot 3^n < m' < 3^{n+1} = f\left(2 \cdot 3^n\right).$$

Since f is an increasing function,

$$f\left(3^n + m\right) = 2 \cdot 3^n + m,$$

for $0 \le m \le 3^n$. Therefore

$$f\left(2 \cdot 3^n + m\right) = f\left(f\left(3^n + m\right)\right) = 3\left(3^n + m\right)$$

for $0 \le m \le 3^n$. Hence

$$f(2001) = f\left(2 \cdot 3^6 + 543\right) = 3\left(3^6 + 543\right) = 3816. \qquad \blacksquare$$

Solution 2: (Sketch) Andrew Dudzik's insight was to recognize that $f(n)$ deals in a very simple way, with the base-3 representation of n. Let $n = a_1 a_n \cdots a_t$ be the base-3 digits of n. For example, if $n = 50$ in base 10, we would write $n = 1212$ in base 3 and hence $a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 2$. Dudzik proved the following:

1. If $a_1 = 1$, then $f(n) = 2a_2a_3 \cdots a_t$.
2. If $a_1 = 2$, then $f(n) = 1a_2a_3 \cdots a_t0$

These two statements can be proven easily with induction; we leave this an an exercise for the reader. Note that Dudzik's formulas allow us to immediately compute $f(2001)$. In base-3 notation, 2001 is equal to 2202010. Thus, by formula #2,

$$f(2202101) = 12020100 = 1 \cdot 3^2 + 2 \cdot 3^4 + 2 \cdot 3^6 + 3^7,$$

and this equals 3816 in base-10 notation. $\blacksquare$

4 A kingdom consists of 12 cities located on a one-way circular road. A magician comes on the 13th of every month to cast spells. He starts at the city which was the 5th down the road from the one that he started at during the last month (for example, if the cities are numbered 1–12 clockwise, and the direction of travel is clockwise, and he started at city #9 last month, he will start at city #2 this month). At each city that he visits, the magician casts a spell if the city is not already under the spell, and then moves on to the next city. If he arrives at a city which is already under the spell, then he removes the spell from this city, and leaves the kingdom until the next month. Last Thanksgiving the capital city was free of the spell. Prove that it will be free of the spell this Thanksgiving as well.

Solution 1 (sketch): Number the cities from 0 to 11. Encode the current state of affairs by starting at city 11, writing "1" if the city is not under a spell, and a "0" if it is under a spell. Do the same thing for each city, from city 10 down to city 0. Interpret the resulting list as the binary expansion of an integer X between 0 and $m = 2^{12} - 1$.

Some thought (using modular arithmetic modulo m, and the way that addition works in base 2) shows that the rules can be interpreted as saying that if the magician starts at city k then 2^k is added to Y, where Y encodes the state as above. After 12 months the magician starts once at all cities, and this has the net effect of adding

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^{11} = 2^{12} - 1 = m$$

to the original X which obviously leaves the state unchanged. $\blacksquare$

Solution 2: We shall denote the state of the kingdom by a 12-tuple of 0s and 1s, with 0 and 1 meaning "free of spell" and "under spell," respectively. For example, suppose that the initial state is (0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0). This means that city #1 is free of the spell, city #2 is under the spell, city #3 is free, etc. Next, define S_j, for $j = 1, 2, \ldots, 12$ to be the transformation of the kingdom after the magician makes a visit that starts at city j. For example,

$$S_1(0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0) = (1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0),$$

since she puts a spell on city #1, then comes to city #2 which is already under the spell, frees it, and stops. Likewise,

$$S_8(0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0) = (1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1),$$

because she puts cities #8–12 under spell, then continues around the circle to #1, puts it under spell, then finally stops after freeing city #2. One more example:

$$S_3(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) = (1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

because she puts every city under the spell, starting with #3, until she comes to a city that is already under the spell, and this happens when she comes to city #3 for the second time.

The key to this problem is the remarkable fact that the S_j transformations commute; i.e.

$$S_iS_j = S_jS_i \quad \text{for all } i, j \in \{1, 2, 3, \ldots, 12\}.$$

Let us assume this fact, to see how it quickly solves the problem. Without loss of generality, suppose the magician starts a visit at city #1. Then during the course of the year, her subsequent visits will start at the following cities, in this order:

$$6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8.$$

In other words, during the year she will make 12 visits, and each visit will start at a different city.[1] In other words, the kingdom will be transformed, in order, by $S_1, S_6, S_{11}, \ldots$. But since these transformations are commutative, we can rearrange the order that we perform the transformations. So, during the course of the year, the kingdom will be transformed by $S_1, S_2, S_3, \ldots, S_{12}$, where we may choose the order in any way that we like.

Consider the state of the kingdom last Thanksgiving. Suppose cities $a, b, c, \ldots$ are spell-free (where city a is the capital) and cities $A, B, \ldots$ are under the spell. Now we shall perform the transformations, in this order:

- $S_A, S_B, \ldots$: Each of these are "single-city" transformations that merely free the starting city and then stop. In other words, after these transformations have been performed, all the cities which were under the spell are now free, and hence at this point, all cities in the kingdom are free. (It may be that no cities were under the spell in the first place, so this step may not take place.)
- S_a: Starting with a, all cities get put under the spell (since they were previously all free), until the magician returns to a, frees it, and stops. Now all cities are under the spell, except for the capital city a.
- The remaining transformations $S_b, S_c \ldots$: Since all cities except a are under the spell, each of these are again "single-city" transformations which end up freeing cities $b, c, \ldots$.

Thus, after the 12 visits, city a is still free. In fact, we have shown more, namely that the state of each of the 12 cities of the kingdom will repeat every 12 months. For example, if city #5 was free last Thanksgiving, it will be free this Thanksgiving; if city #7 was under the spell last Thanksgiving, it will also be under the spell this Thanksgiving, etc.

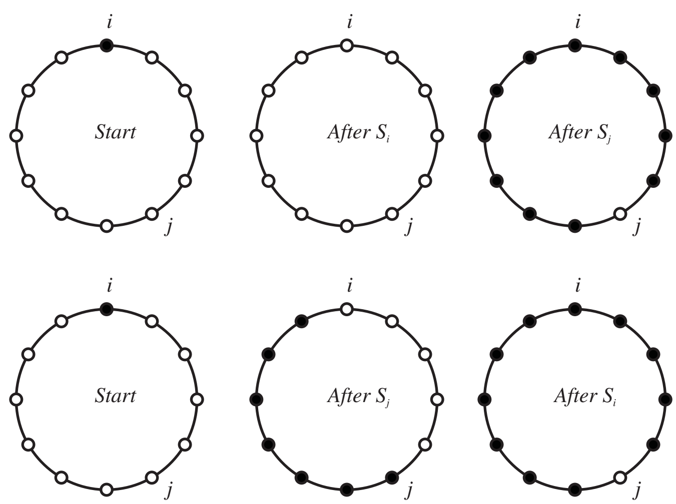
[1] The sophisticated reader will observe that this is a consequence of the fact that 5 is relatively prime to 12.

It remains to prove the key fact that the S_j transformations are commutative. Specifically, we will show that

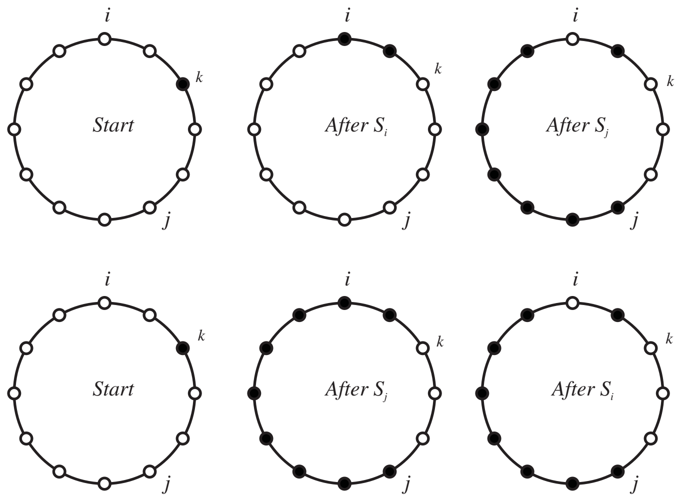
$$S_j(S_i(x_1, x_2, \ldots, x_{12})) = S_i(S_j(x_1, x_2, \ldots, x_{12}))$$

for any distinct i, j and for any 12-tuple $(x_1, x_2, \ldots, x_{12})$. We can do this by examining several cases which depend on the 12-tuple.

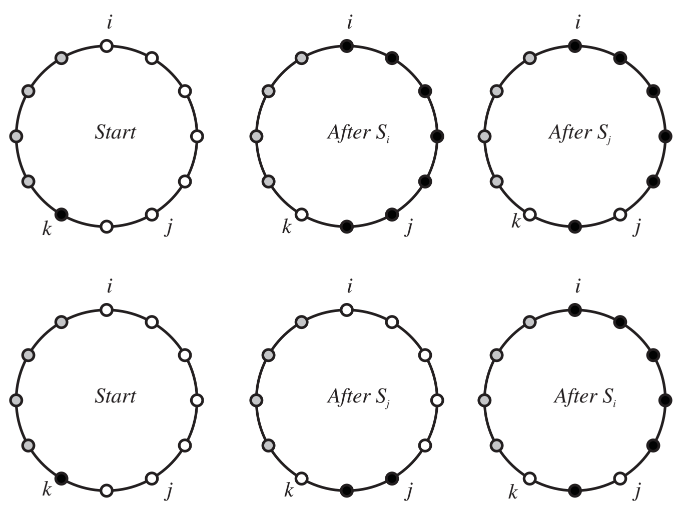
1. All the coordinates are zero. In other words, all towns are currently free of the spell. It is easy to check that performing S_i on this will put make all $x_k = 1$ except $x_i = 0$. Then performing S_j will change x_j to 0. The net result is that all values are 1 except for $x_i = x_j = 0$. Clearly, we will have the same result if we perform S_j first, and then S_i.
2. Exactly one of the coordinates is 1.
   (a) Suppose $x_i = 1$ and all others are zero. Performing S_i makes $x_i = 0$ and hence now all values are zero. Then S_j results in all values equalling 1 except $x_j = 0$. Conversely, performing S_j first will put all cities from j to $i-1$ under the spell (travelling clockwise) until the magician reaches city i which she then frees of the spell and stops. So now there is an "arc" of 1's from x_j to x_{i-1} and all other values are 0. Performing S_i on this turns the values of the arc from i to $j-1$ into 1's, until she reaches city j, and then she frees this city and stops. The net result again is that all values are 1 except $x_j = 0$. The figure below illustrates this. We assume clockwise travel and denote free/spell by white/black, respectively.



   (b) Suppose $x_j = 1$ and all others are zero. This case is exactly the same as 2(a) above; just interchange i and j.
   (c) Suppose that $x_k = 1$ and all others are zero, with $k \neq i$ and $k \neq j$. The figure below illustrates the case where k lies on the clockwise arc between i and j. The case where k lies between j and i has a similar picture.



3. At least two coordinates equal 1.
   (a) Suppose $x_k = x_\ell = 1$, where k lies on the clockwise arc from i to j and ℓ lies on the arc from j to i (it is possible that k or ℓ may equal one or both of i and j.) Then the order of S_i and S_j will not matter—each transformation alters non-overlapping arcs: S_i only affects the arc from i to k, while S_j affects the arc from j to ℓ.
   (b) One of the arcs described in 3(a) contains only zeros. Without loss of generality, suppose that the arc from i to j is all zeros (including i, j) and let k be the first coordinate clockwise from j such that $x_k = 1$. The picture below handles this case. The cities between k and i are not shaded, but they could be black or white; their state is not relevant, since they will be unaffected by S_i or S_j.



These cases encompass all possibilities, so we conclude that S_i and S_j commute for all 12-tuples and the proof is complete. ■

REMARK: There is a third solution method, which examines the number of times each city's state is changed during the course of the year.

5 For each positive integer n, let a_n be the number of permutations τ of $\{1, 2, \ldots, n\}$ such that $\tau(\tau(\tau(x))) = x$ for $x = 1, 2, \ldots, n$. The first few values are

$$a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 9.$$

Prove that 3^{334} divides a_{2001}.

(A **permutation** of $\{1, 2, \ldots, n\}$ is a rearrangement of the numbers $\{1, 2, \ldots, n\}$, or equivalently, a one-to-one and onto function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. For example, one permutation of $\{1, 2, 3\}$ is the rearrangement $\{2, 1, 3\}$, which is equivalent to the function $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$.)

Solution: Consider the permutations τ of $\{1, 2, \ldots, n\}$ such that $\tau(\tau(\tau(x))) = x$ for $x = 1, 2, \ldots, n$. Then for each $x \in \{1, 2, \ldots, n\}$, there are only two possibilities. Either x is a a **fixed point**; i.e., $\tau(x) = x$, or else x is a member of a **3-cycle** (xyz); i.e. $\tau(x) = y, \tau(y) = z, \tau(z) = x$, where x, y, z are 3 distinct numbers.

We can thus partition these permutations into two cases: Either 1 is a fixed point, or it is not. In the first case, the remaining elements $\{2, \ldots, n\}$ can permuted in a_{n-1} ways. In the second case, 1 is part of a 3-cycle $(1ij)$, where $i \neq j$ and $i, j \in \{2, \ldots, n\}$. There are $(n-1)(n-2)$ such 3-cycles (order counts). Then the remaining $n - 3$ elements can be permuted in a_{n-3} ways. Hence we have (for $n > 3$)

$$a_n = a_{n-1} + (n-1)(n-2)a_{n-3}. \tag{1}$$

Define b_n to be the highest power of 3 which divides a_n; i.e.

$$3^{b_n} \| a_n.$$

If $n \not\equiv 0 \pmod 3$, then $(n-1)(n-2)$ will be a multiple of 3, so equation (1) yields $a_n = a_{n-1} + 3ka_{n-3}$, where k is a positive integer. Hence

$$b_n \geq \min(b_{n-1}, b_{n-3} + 1), \quad n \not\equiv 0 \pmod 3. \tag{2}$$

If $n \equiv 0 \pmod 3$, then $(n-1)(n-2) \equiv -1 \pmod 3$, but both $(n-2)(n-3)$ and $(n-3)(n-4)$ are multiples of 3. Plugging into (1), we have

$$\begin{aligned} a_n - a_{n-1} &= (n-1)(n-2)a_{n-3} = (3u-1)a_{n-3}, \\ a_{n-1} - a_{n-2} &= (n-2)(n-3)a_{n-4} = 3va_{n-4}, \\ a_{n-2} - a_{n-3} &= (n-3)(n-4)a_{n-5} = 3wa_{n-5}, \end{aligned}$$

for positive integers u, v, w. Adding these, we get

$$a_n - a_{n-3} = (3u-1)a_{n-3} + 3va_{n-4} + 3wa_{n-5},$$

so

$$a_n = 3(ua_{n-3} + va_{n-4} + wa_{n-5}).$$

This implies that

$$b_n \geq 1 + \min(b_{n-3}, b_{n-4}, b_{n-5}), \quad n \equiv 0 \pmod 3, n > 5. \tag{3}$$

We know that $b_1 = 0, b_2 = 0, b_3 = 1$. Employing (2), we get $b_4 \geq 1, b_5 \geq 1$. Then (3) yields $b_6 \geq 1$. Applying (2) again yields $b_7 \geq 1, b_8 \geq 1$. But now when (3) is applied, we have $b_9 \geq 2$. It is evident that continuing this process, (2) and (3) will increment b_n by at least one as n increases by 6. In other words,

$$b_{6k+3} \geq k + 1.$$

Since $2001 = 6 \cdot 333 + 3$, we are done. ■

REMARK: The above inequalities are not the best. Direct computation using the recurrence (1) shows that a_{2001} is a 3,830-digit number and that $3^{445} \| a_{2001}$.