5th Bay Area Mathematical Olympiad

February 25, 2003

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 An integer is a *perfect* number if and only if it is equal to the sum of all of its divisors except itself. For example, 28 is a perfect number since $28 = 1 + 2 + 4 + 7 + 14$.

Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$, where n is a positive integer. An integer is a *factorial* if and only if it is equal to $n!$ for some positive integer n. For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.

Find all perfect numbers greater than 1 that are also factorials.

2 Five mathematicians find a bag of 100 gold coins in a room. They agree to split up the coins according to the following plan:

- The oldest person in the room proposes a division of the coins among those present. (No coin may be split.) Then all present, including the proposer, vote on the proposal.
- If at least 50% of those present vote in favor of the proposal, the coins are distributed accordingly and everyone goes home. (In particular, a proposal wins on a tie vote.)
- If fewer than 50% of those present vote in favor of the proposal, the proposer must leave the room, receiving no coins. Then the process is repeated: the oldest person remaining proposes a division, and so on.
- There is no communication or discussion of any kind allowed, other than what is needed for the proposer to state his or her proposal, and the voters to cast their vote.

Assume that each person is equally intelligent and each behaves optimally to maximize his or her share. How much will each person get?

3 A *lattice point* is a point (x, y) with both x and y integers. Find, with proof, the smallest n such that every set of n lattice points contains three points that are the vertices of a triangle with integer area. (The triangle may be *degenerate*, in other words, the three points may lie on a straight line and hence form a triangle with area zero.)

Please turn over for problems #4 and #5.

4 An integer $n > 1$ has the following property: for every (positive) divisor d of n, $d + 1$ is a divisor of $n + 1$. Prove that n is prime.

5 Let $ABCD$ be a square, and let E be an internal point on side AD. Let F be the foot of the perpendicular from B to CE. Suppose G is a point such that $BG = FG$, and the line through G parallel to BC passes through the midpoint of EF. Prove that $AC < 2 \cdot FG$.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2003 Awards Ceremony**, which will be held at Stanford University from 11–2 on Sunday, March 9. This event will include lunch (free of charge), a mathematical talk by Professor Ravi Vakil of Stanford University, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.

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Problems and Solutions

1 An integer is a *perfect* number if and only if it is equal to the sum of all of its divisors except itself. For example, 28 is a perfect number since $28 = 1 + 2 + 4 + 7 + 14$.

Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$, where n is a positive integer. An integer is a *factorial* if and only if it is equal to $n!$ for some positive integer n. For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.

Find all perfect numbers greater than 1 that are also factorials.

Solution: The only perfect factorial is $6 = 3!$. Certainly, $2! = 2$ is not perfect. For $n > 3$, note that $n! = 6k$, where $k > 1$, and thus the factors of $n!$ will include $1, k, 2k, 3k$. This sums to $6k + 1$, showing that $n!$ is not perfect. ■

2 Five mathematicians find a bag of 100 gold coins in a room. They agree to split up the coins according to the following plan:

- The oldest person in the room proposes a division of the coins among those present. (No coin may be split.) Then all present, including the proposer, vote on the proposal.
- If at least 50% of those present vote in favor of the proposal, the coins are distributed accordingly and everyone goes home. (In particular, a proposal wins on a tie vote.)
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- There is no communication or discussion of any kind allowed other than what is needed for the proposer to state his or her proposal, and the voters to cast their vote.

Assume that each person wishes to maximize his or her share of the coins and behaves optimally. How much will each person get?

Solution: The first (oldest) person will get 98 coins, the second will get no coins, and one coin apiece will go to two of the last three people. If we write the distribution as an ordered 5-tuple, with leftmost being oldest, etc., then the three possible solutions are $(98, 0, 1, 0, 1)$, $(90, 0, 0, 1, 1)$, and $(98, 0, 1, 1, 0)$.

To see why, we first mention a few principles. Since the mathematicians are all intelligent, and equally intelligent, but only out for themselves and unable to cooperate or negotiate, each person must assume the worst. No one will approve a plan that gives them zero coins. But furthermore, no one will vote *against* a plan that gives them a positive number of coins, if it may be *possible* that they could end up with fewer coins. This is not a game of chance; a person will settle, for example, for just one coin, if by rejecting this offer he or she may end up with no coins.

This explains why the common wrong answer "everyone votes against the oldest in order to decrease the number of players" will not happen with rational players. If the number of players dropped to 4 people, then the first person would have the advantage, and need only propose something that one other person would agree to. For example, the proposal may be $(50, 0, 0, 50)$. This would give players #2, #3 (originally #3,#4) no coins at all, but only 2 votes are needed for the proposal to pass. On the other hand, the proposal could be $(50, 0, 50, 0)$. In other words, when there are 5 people present, it is *not* in the best interest of #3 ,#4, or #5 to reject the proposal, if they are offered non-zero amounts of coins, since they *may* end up with zero later, and have absolutely no control over what happens!

Indeed, by analyzing smaller cases, it becomes clear how much advantage the first person has, and how the second person is in the worst situation. With just two people, $(100, 0)$ wins all for the first person. With three people, $(99, 0, 1)$ will be accepted, since #3 dare not reject it, for then #3 would be the loser in the 2-person situation. Consequently, with 4 people, the winning proposals will be $(99, 0, 1, 0)$ or $(99, 0, 0, 1)$. In both cases, the people with zero will vote against the proposal, and the one person who is offered a single coin will accept, since in no circumstances will they do better than this. (Conceivably, the youngest person can vote against $(99, 0, 0, 1)$, reasoning that the next proposal will be $(99, 0, 1)$ which also offers him or her one coin; knowing this, the most likely 4-person proposal will be $(99, 0, 1, 0)$ which will definitely not be rejected by person #3.)

Finally, looking at 5 people, it is now clear that the first person need only "pay off" two others, with the least amount. Person #2 cannot be paid off, since he or she will always be better off being the proposer. Knowing this, person #1 always offers nothing to person #2. Persons #3, #4, #5 all realize that they may end up with no coins at all, by the analysis above. Consequently, if any two of them are offered just a single coin, they will accept, realizing that failing to accept will result in just 4 players, with absolutely no guarantee of any coins at all. In particular, player #3 (in the 5-person situation) realizes that with 4 players, he will surely be offered (and will receive) no coins, so she will gladly accept one coin in the 5-person situation. Persons #4 and #5 (in the 5-person case) are in the situation of not knowing for sure whether they will be offered 1 or 0 coins in the 4-person case. Consequently, if either of them is offered just one coin, the offer is accepted, since a certainly of getting one coin is superior to a possibility of getting no coins. ■

3 A *lattice point* is a point (x, y) with both x and y integers. Find, with proof, the smallest n such that every set of n lattice points contains three points that are the vertices of a triangle with integer area. (The triangle may be *degenerate*, in other words, the three points may lie on a straight line and hence form a triangle with area zero.)

Solution: Clearly, $n = 4$ is too small, for the 4 points could be vertices of a unit square, and all the possible triangles will have areas of $1/2$. We will show that $n = 5$ works. Every lattice point falls into one of four parity classes: (even, even), (even, odd), (odd, even), and (odd, odd). For example the point $(9, 8)$ is in the (odd, even) class. By the pigeonhole principle, among the 5 lattice points there will be at least two points, A and B, in the same parity class. We claim that if C is any other lattice point, the triangle ABC will have integral area.

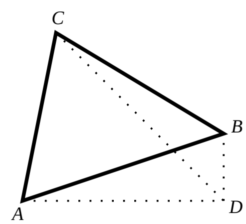
We will use the following simple lemma:

> *If the vertices of a triangle are all lattice points, and one side is horizontal or vertical with even length, then the triangle must have integer area.*

To see why the lemma is true, let PQR be the triangle with PQ even, and, say, horizontal. Let h be the length of the altitude from R to PQ. Clearly, h is an integer, since it is equal to the absolute value of the difference in y-coordinates of R and P. Then $[PQR] = PQ \cdot h/2$, and the lemma follows. (We are using the notation $[PQR]$ to denote the area of triangle PQR.)

Now we will show that $[ABC]$ must be an integer. If AB is horizontal or vertical, then the length of AB is even (since the endpoints are in the same parity class), so the lemma immediately implies that $[ABC]$ is an integer.

If AB is neither horizontal nor vertical, choose the lattice point D such that ADB is a right triangle with right angle at D (there may be more than one choice).



Then

$$[ABC] = [ACBD] - [ADB].$$

Since AD and DB both have even length, $[ABC]$ is an integer. Furthermore,

$$[ACBD] = [CAD] + [CDB],$$

and by the same reasoning $[CAD]$ and $[CDB]$ are both integers. ■

4 An integer $n > 1$ has the following property: for every (positive) divisor d of n, $d + 1$ is a divisor of $n + 1$. Prove that n is prime.

Solution: Let p be the smallest prime factor of n, and let $d = n/p$. Then,

$$\frac{np + p}{n + p} = \frac{p(n + 1)}{p(d + 1)} = \frac{n + 1}{d + 1}$$

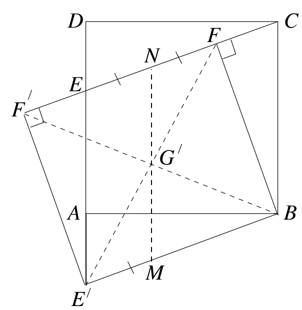
is an integer, by assumption. But $n + p$ also divides $np + p^2$, so it must divide the difference $(np + p^2) - (np + p) = p^2 - p$. In particular, $n + p \leq p^2 - p$, since the latter is positive.

This certainly gives $n < p^2$, so, dividing by p, we have $d < p$. Suppose that d has some prime factor q; then $q \leq d < p$. On the other hand, q also divides n, and then the minimality of p gives $q \geq p$. This is a contradiction, so q cannot exist, and we conclude that $d = 1$. Then, $n = p$, as needed. ■

5 Let $ABCD$ be a square, and let E be an internal point on side AD. Let F be the foot of the perpendicular from B to CE. Suppose G is a point such that $BG = FG$, and the line through G parallel to BC passes through the midpoint of EF. Prove that $AC < 2 \cdot FG$.

Solution 1: First note that, for given E, F, there is only one point G with the required properties: since $BG = FG$, G must lie on the perpendicular bisector of BF, and by the second condition, G lies on the line through the midpoint of EF parallel to BC; G must thus be the unique intersection of these two lines. We now give an alternative construction for G from which the result will follow.

Since BA, CD are parallel and equal, we may translate $\triangle CDE$ to give a triangle $\triangle BAE'$. Then E', A, E are collinear, with $E'E = E'A + AE = ED + AE = AD = BC$, and $E'E$, BC are parallel, so we may also translate $\triangle BCF$ to give $\triangle E'EF'$. Then F, E, F' are collinear. Now $E'F' = BF$, and $\angle E'F'E = \angle BFC = 90^\circ = \angle BFE$; hence, $BE'F'F$ is a rectangle. Let G' be its center. Certainly $BG' = FG'$. Let the line through G' parallel to BC hit BE', FF' at M, N respectively; then symmetry gives $E'M = FN$. However, by translation, $EN = E'M$, so $EN = FN$, and the parallel to BC through G' bisects EF. Thus, by the uniqueness of G already proven, $G = G'$, the center of $BE'F'F$.



Now,

$$2 \cdot FG = FE' = \sqrt{BF^2 + BE'^2} > \sqrt{2 \cdot BF \cdot BE'}$$

(by AM-GM, strict because $BF < BC = CD < CE = BE'$)

$$= \sqrt{2[BFF'E']} = \sqrt{2([BAE'] + [E'EF'] + [ABFE]}$$

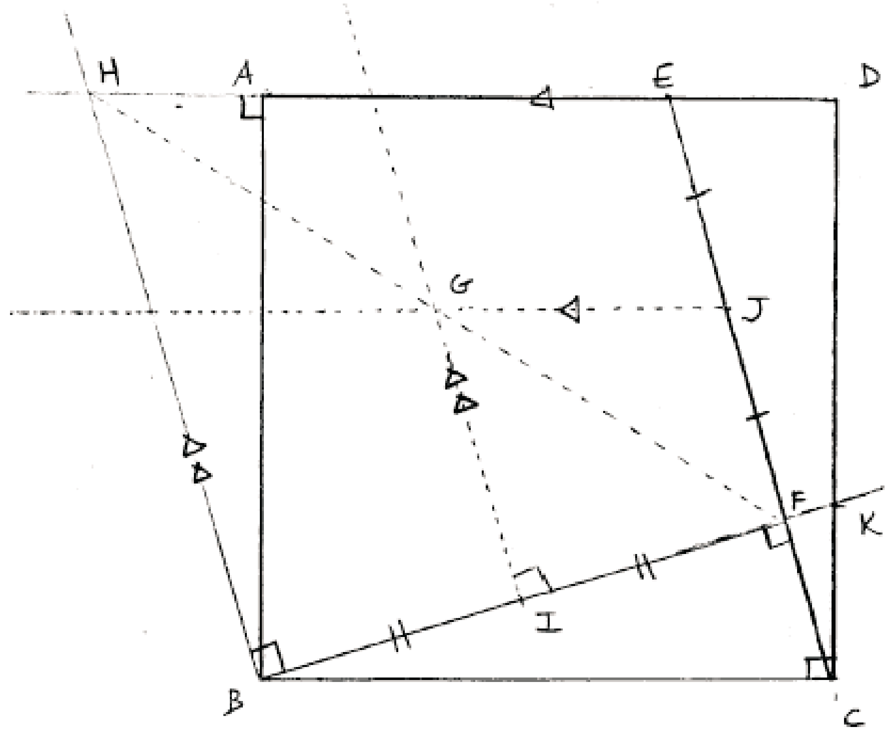
(here brackets denote areas)

$$= \sqrt{2([CDE] + [BCF] + [ABFE])} = \sqrt{2[ABCD]}$$

$$= \sqrt{2} \cdot AB = AC. \qquad \blacksquare$$

Solution 2: The following solution, due to Philip Sung of Saratoga High School, not only establishes the inequality, but computes the difference.

Not only that, Philip's handwriting and artwork is so good that the lazy writer of these solutions can photocopy his work below! Study carefully this beautiful solution, with its nice ideas of auxilliary constructions and dilation ("scaling").



Let I bisect $\overline{BF}$ and J bisect $\overline{EF}$. Extend AD, and draw a parallel to $\overline{FE}$ through B; let this meet AD at H. Extend $\overline{BF}$ so it meets DC at K.

We will show that FBHE is the image of FIGJ under a scaling with ratio 2 and center F. Obviously, F is taken to F, I is taken to B, and J to E. G is the intersection of IG and JG, so it is mapped to the intersection of their images. $\overline{BH}$ is the image of $\overline{IG}$ because it is parallel to $\overline{IG}$ and passes through the image of I (i.e., B.) Similarly, $\overline{EH}$ is the image of $\overline{JG}$. H lies on both $\overline{EH}$ and $\overline{BH}$, so it is the image of G.

$\angle HBF$ is right, as is $\angle ABC$. So they cut off equal angles: $\angle HBA \cong \angle KBC$. In addition, because $\overline{AB} \cong \overline{CB}$ and $\angle HAB \cong \angle KCB$ (they are both right), $\triangle HAB \cong \triangle KCB$ and $\overline{AH} \cong \overline{CK}$.

FH = 2 FG because H is the image of G under aforementioned dilation. The problem statement then reduces to:

$$AC < FH.$$

We will show, equivalently, that $AC^2 < FH^2$ (AC and FH are both positive.)

$$
\begin{aligned}
FH^2 &= FB^2 + BH^2 && (\triangle FBH \text{ is right}) \\
&= FB^2 + BA^2 + AH^2 && (\triangle BAH \text{ is right}) \\
&= FB^2 + BA^2 + CK^2 && (AH = CK) \\
&= FB^2 + BA^2 + CF^2 + FK^2 && (\triangle CFK \text{ is right}) \\
&= (FB^2 + CF^2) + BA^2 + FK^2 && \\
&= BC^2 + BA^2 + FK^2 && (\triangle BFC \text{ is right}) \\
&= AC^2 + FK^2 && (\triangle ABC \text{ is right}) \\
&> AC^2 && (FK \text{ is nonzero})
\end{aligned}
$$

$$FH^2 > AC^2$$

and we're done!

$\therefore$