6th Bay Area Mathematical Olympiad

February 24, 2004

Problems and Solutions

1 A *tiling* of the plane with polygons consists of placing the polygons in the plane so that interiors of polygons do not overlap, each vertex of one polygon coincides with a vertex of another polygon, and no point of the plane is left uncovered. A *unit* polygon is a polygon with all sides of length one.

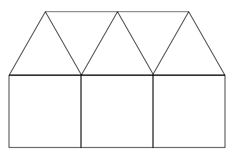
It is quite easy to tile the plane with infinitely many unit squares. Likewise, it is easy to tile the plane with infinitely many unit equilateral triangles.

(a) Prove that there is a tiling of the plane with infinitely many unit squares and infinitely many unit equilateral triangles in the same tiling.

(b) Prove that it is impossible to find a tiling of the plane with infinitely many unit squares and finitely many (and at least one) unit equilateral triangles in the same tiling.

Solution:

(a) This can be done easily with parallel rows of squares and triangles, as shown.



Of course, other tilings are possible.

(b) Many people received only partial credit for part (b), because their arguments were not rigorous. In order to show that a tiling is not possible, you need a completely general argument, that handles all cases. Since there are infinitely many cases, this can be problematical. The way out is a very neat idea known as the *Extreme Principle*, which essentially says, "focus on the largest or smallest entity." The advantage of this approach is that we now are reduced to studying just one polygon in the infinite plane.

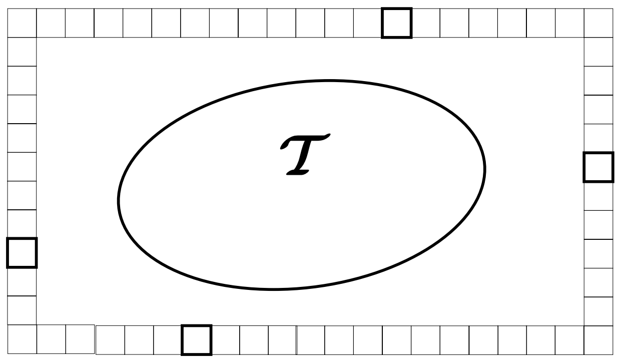
Method 1: Suppose, to the contrary, that there were such a tiling. Since there are only finitely many triangles, there is a vertex that is "northenmost." If there are ties, pick the vertex that lies furthest to the "east." At most one other triangle (located to the west) can share this vertex. This is a contradiction, since the only way that triangles and squares can share a vertex in a tiling is with 3 triangles and two squares (in order to add up to 360 degrees).

Method 2:

Suppose, to the contrary, that there were such a tiling. Since there are only finitely many triangles, they are contained in a bounded region . Let N be a square that lies to the "north" of (i.e., its south side is further north than any point in). Likewise, let E, W, S be squares that lie, respectively, east, west, and south of .

Observe that the east and west neighbors of N must be squares, since there are no triangles that far north. Thus, there is an infinite east-west chain of connected squares containing N. Likewise, there is an infinite east-west chain of squares containing S, and there are two infinite north-south chains, one containing W, and one containing E.

These four chains meet, forming a rectangular boundary of connected squares that completely encloses . Now we have a contradiction: This rectangular region has rational (in fact, integer) area, yet it is tiled with a non-zero number of equilateral triangles, plus, perhaps, some squares. But the area of each triangle is $\sqrt{3}/4$, and hence the area of the entire collection of triangles and squares inside this rectangular region is irrational.



Method 3: Suppose that there were such a tiling. By similar reasoning to method 2, we deduce that the tiling is "eventually" all squares; thus it can be formed by starting with an all-squares tiling, and then removing a finite number of squares, and filling in the "holes" with a finite number of triangles. The "holes" only have 90-degree angles; they can never be filled with triangles, since 90 is not a multiple of 60.

2 A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

Solution: The following construction works:

1. Draw a line from P to A, intersecting the circle at C.
2. Draw a line from P to B, intersecting the circle at D.
3. Draw lines AD and BC, and let E be their point of intersection.
4. Draw a line from P through E; this will be the desired perpendicular line.

This works because $AD \perp PB$ and $BC \perp PA$; hence AD and BC are altitudes of triangle APB. It is well known that the three altitudes of a triangle intersect in a point, so E is the intersection of all three altitudes. It follows that the line through PE is an altitude.

3 NASA has proposed populating Mars with 2,004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing

N tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of N that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

Solution: The problem is equivalent, in general, to finding the least number of edges required so that a graph on n vertices will be connected, i.e., one can reach any vertex from any other vertex by following the edges of the graph. (We are letting settlements be vertices and tunnels be edges, of course).[1] This value is $\binom{n-1}{2}+1$.

Here $\binom{m}{2}$ counts the number of all possible pairs in a group of m people, or equivalently, the number of edges in a graph with m vertices where every two vertices are connected with an edge. This latter graph is called a complete graph on m vertices.

To see that the minimum number of edges must be $\binom{n-1}{2}+1$, we first observe that it cannot be less than this, since $n-1$ vertices can be connected to one another with $\binom{n-1}{2}$ edges, leaving the nth vertex isolated.

Next we will show that $\binom{n-1}{2}+1$ edges will guarantee that the graph is connected.

Method 1: Suppose to the contrary, that the graph is not connected. Then it consists of k connected components, each containing $v_1, v_2, \ldots, v_k$ vertices. Each component has at most $\binom{v_i}{2}$ edges. We claim that

$$\binom{v_1}{2}+\binom{v_2}{2}+\cdots+\binom{v_k}{2} \le \binom{n-1}{2},$$

which establishes the contradiction.

The above inequality an easy consequence of the two-term inequality

$$\binom{a}{2}+\binom{b}{2} \le \binom{a+b-1}{2},$$

which can be established by considering a complete graph on a vertices (with $\binom{a}{2}$ edges) and a complete graph on b vertices (with $\binom{b}{2}$ edges, and then "gluing" them together on one vertex. This produces a new graph with $a+b-1$ vertices which must have at most $\binom{a+b-1}{2}$ edges.

Method 2: Since there are at most $\binom{n}{2}$ tunnels possible, there will be at most

$$\binom{n}{2}-\left(\binom{n-1}{2}+1\right)=n-2$$

tunnels that are not drawn. Call these "antitunnels." Suppose to the contrary, that the graph is not connected. Then two settlements, A and B will not be connected. Thus, A and B are joined by an antitunnel. Furthermore, for each settlement X that is neither A nor B, there can be no path drawn from A to X and then from X to B. In other words, at least one of the connections AX or XB must be an antitunnel. However, this would require $n-2$ antitunnels, in addition to the antitunnel joining A and B. Thus $n-1$ antitunnels are needed, but at most $n-2$ are available; a contradiction.

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[1]For introductory information about graph theory, there are many good books. See, for example, *Pearls of Graph Theory* by Hartsfield and Ringel.

4 Suppose one is given n real numbers, not all zero, but such that their sum is zero. Prove that one can label these numbers $a_1, a_2, \ldots, a_n$ in such a manner that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 < 0.$$

Solution: Let the given numbers (in an arbitrary order) be $b_1, b_2, \ldots, b_n$. For every possible permutation π of $\{1, 2, \ldots, n\}$, consider the sum

$$b_{\pi(1)}b_{\pi(2)} + b_{\pi(2)}b_{\pi(3)} + \cdots + b_{\pi(n-1)}b_{\pi(n)} + b_{\pi(n)}b_{\pi(1)}.$$

We wish to show that some such sum is negative, so assume otherwise. For every two distinct elements $i, j \in \{1, \ldots, n\}$, the term b_ib_j appears N times among these sums, where N does not depend on i, j, by symmetry. (In fact, one can show that $N = n(n-2)!$.) Each such sum is assumed to be nonnegative; adding these inequalities for all permutations π, and dividing by N, we have

$$\sum_{i \neq j} b_ib_j \geq 0.$$

However, we also know that $\sum_i b_i^2 > 0$ (strictly, since not all b_i are zero). Thus

$$(b_1 + \cdots + b_n)^2 = \sum_i b_i^2 + \sum_{i \neq j} b_ib_j > 0.$$

But since $b_1 + \cdots + b_n = 0$, we have a contradiction. So our assumption was false, and the needed negative sum does exist.

5 Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.

1. $f(0) = 2004$.
2. If x is irrational, then $f(x)$ is also irrational.

(Notes: A polynomial is *monic* if its highest degree term has coefficient 1. Thus, $f(x) = x^4 - 5x^3 - 4x + 7$ is an example of a monic polynomial with integer coefficients.

A number x is rational if it can be written as a fraction of two integers. A number x is irrational if it is a real number which cannot be written as a fraction of two integers. For example, $2/5$ and -9 are rational, while $\sqrt{2}$ and π are well known to be irrational.)

Solution: The polynomial $x + 2004$ certainly meets the two conditions. In fact, this is the only one. We will prove this using three ingredients: the infinitude of primes, the Rational Roots Theorem for polynomials, and the approximation principle that x^n dominates any polynomial of lower degree, for large enough x.

Note that the only monic constant polynomial is $f(x) = 1$, which fails Condition 1; and that the only monic degree 1 polynomial satisfying Condition 1 is $f(x) = x + 2004$. Thus, we need to eliminate all polynomials of degree 2 or more. To this end, it is sufficient to show that, given any monic polynomial

$f(x)$ with integer coefficients of degree 2 or more, there exists an integer a such that $f(x)+a=0$ has an irrational solution x (for then $f(x)=-a$ is rational for an irrational number x).

Let $f(x)=x^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0$ be a polynomial with integer coefficients, with $n\geq 2$. It may be the case that $f(x)$ has no real roots, for example, if n is even and the graph of $y=f(x)$ lies above the x-axis. But certainly, if a is a sufficiently large negative integer, we can guarantee that $f(x)+a=0$ will have at least one real solution. In fact, by further making a a larger negative number, we can ensure that, say, the largest of the solutions of $f(x)+a=0$ has absolute value bigger than 1: $|x|>1$.

Moreover, regardless of how large a negative number a needs to be, we can choose a so that $c_0+a=-p$ where p is prime. This is because there are infinitely many prime numbers. Now we can apply the Rational Roots Theorem, according to which all rational solutions $\frac{r}{s}$ of the monic integer coefficient polynomial $f(x)+a$ must satisfy: s divides the leading coefficient of $f(x)$ and r divides the last (free term) of $f(x)$; in other words, s divides 1 and r divides p. Since p is prime, this gives only four possible rational solutions: $x=\pm 1,\pm p$. Since we have ensured that $|x|>1$, we are left with $x=\pm p$.

Let $g(x)=f(x)+a$. From the well known inequalities of absolute values $|y+z|\geq|y|-|z|$ and $|y+z|\leq|y|+|z|$, we obtain:

$$|g(x)|=|x^n+c_{n-1}x^{n-1}+...+c_1x-p|\geq|x^n|-|c_{n-1}x^{n-1}+...+c_1x-p|$$

and as long as $|x|>1$ and $n\geq 2$:

$$\begin{aligned}|c_{n-1}x^{n-1}+\cdots+c_1x-p| &\leq |c_{n-1}||x|^{n-1}+\cdots+|c_1||x|+p\\ &\leq (|c_{n-1}|+\cdots+|c_1|)|x|^{n-1}+p^{n-1}.\end{aligned}$$

If we let $S=|c_{n-1}|+\cdots+|c_1|$, we can put everything together:

$$|g(\pm p)|\geq p^n-(S+1)p^{n-1}=p^{n-1}\big(p-(S+1)\big).$$

Since S is fixed, we can choose the prime p large enough so that $p>S+1$, and hence the quantity $p-(S+1)$ is positive. Therefore, $g(\pm p)\neq 0$.

Thus $g(x)$ has a real zero x, which cannot be rational since the only possibilities for rational zeros $\pm p$ fail to be zeros by the above. We conclude that x is an irrational root of $g(x)$, whereas $f(x)=-a$ is an integer, hence rational. This contradicts Condition 2, and eliminates all polinomials of degree 2 or more.

Finally, we are left with only one possible solution: $f(x)=x+2004$.