8th Bay Area Mathematical Olympiad

February 28, 2006

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

 (a) Every student must move to a new chair.

 (b) A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules above allow two students in adjacent chairs to exchange places.)

Show that this procedure can be done if n is even, and cannot be done if n is odd.

Solution: If n is even, there are many ways to do this. One is simply to exchange adjacent students in each row: the student in an odd-numbered chair k exchanges places with the student in chair $k+1$. In other words, exchange students in chairs 1 and 2, those in chairs 3 and 4, and so on. Since there are an even number of students in each row, every student in every row will move and the teacher's two conditions can be satisfied.

If n is odd, imagine that the chairs are colored alternately black and white as on a chessboard with a black chair in one corner (and hence all four corner chairs are black). It is easy to see that $(n^2+1)/2$ of the chairs are colored black and $(n^2-1)/2$ are white, so there is one more black than white chair. Any valid rearrangement must move each student to a chair of the opposite color. The conditions cannot be satisfied since there is one more black chair than white chair, so some student seated in a black chair will have nowhere to go. ∎

2 Since $24 = 3+5+7+9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

 (a) Can 2005 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

(b) Can 2006 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

Solution: Let $N = (2k+1) + (2k+3) + \cdots + (2k+2n-1)$ where n and k are integers, $n \geq 2$, $k \geq 0$. Then $N = [(2k+1) + (2k+2n-1)]n/2 = (2k+n)n$, which is a product of two integers with the same parity, since adding the even number $2k$ to the integer n does not change its parity.

1. Since $2005 = 401 \cdot 5$, taking $n = 5$ gives $2k+5 = 401$, i.e. $2k+1 = 397$. Checking, we see that $397 + 399 + 401 + 403 + 405 = 401 \cdot 5 = 2005$, so 2005 *can* be represented as the sum of at least two consecutive odd positive integers.

2. Since $2006 = 2 \cdot 1003 = 2 \cdot 17 \cdot 59$ is divisible by 2 but not by 4, every pair of integers with a product of 2006 consists of two integers with different parity, (i.e. one odd and one even). Therefore 2006 *cannot* be represented as the sum of at least two consecutive odd positive integers.

∎

3 In triangle ABC, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of triangle ABC if and only if P is the centroid of triangle $A_1B_1C_1$.

Note: A median in a triangle is a segment connecting a vertex of the triangle with the midpoint of the opposite side. The centroid of a triangle is the intersection point of the three medians of the triangle. The centroid of a triangle is also known by the names "center of mass" and "medicenter" of the triangle.

Solution 1: Assume first that P is the centroid of $\triangle ABC$. This means that AA_1, BB_1 and CC_1 are the medians of $\triangle ABC$, and hence A_1, B_1 and C_1 are the midpoints of BC, CA and AB, respectively. As a midsegment of $\triangle ABC$, segment B_1A_1 is parallel to AB, and similarly, C_1A_1 is parallel to AC. In the parallelogram $AC_1A_1B_1$, the diagonals AA_1 and B_1C_1 bisect each other. In particular, AA_1 contains the median of $\triangle A_1B_1C_1$ passing through vertex A_1. Similarly, BB_1, respectively CC_1, contains the median of $\triangle A_1B_1C_1$ passing through vertex B_1, respectively C_1. Since AA_1, BB_1 and CC_1 intersect in P and they are extensions of the three medians of $\triangle ABC$, by definition, P is also the centroid of $\triangle A_1B_1C_1$, and we are done with this part.
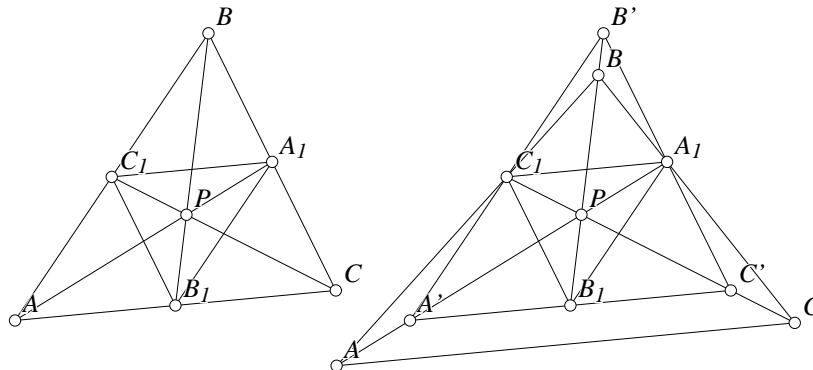
For the other part of the problem, assume that P is the centroid of $\triangle A_1B_1C_1$. We construct $\triangle A'B'C'$ in the following way. Draw a line through A_1 parallel to B_1C_1, another line through B_1 parallel to C_1A_1, and a third line through C_1 parallel to A_1B_1. Let the three lines intersect pairwise in points A', B' and C' so that A_1 lies on $B'C'$, B_1 lies on $C'A'$, and C_1 lies on $A'B'$. From parallelograms $B'A_1B_1C_1$ and $C'B_1C_1A_1$, it follows that $C'A_1 = B_1C_1 = B'A_1$, i.e. A_1 is the midpoint of $B'C'$. Similarly, B_1 and C_1 are the midpoints of $C'A'$ and $A'B'$, respectively. Further, from the same parallelograms follows, for instance, that $C'C_1$

bisects A_1B_1, and hence $C'C_1$ contains the median of $\triangle A_1B_1C_1$ and therefore it passes through point P. Similarly, $B'B_1$ and $A'A_1$ pass through P. Thus, $A'A_1$, $B'B_1$ and $C'C_1$ intersect in point P, and are the medians in $\triangle A'B'C'$. Hence P is the centroid of $\triangle A'B'C'$ too. (Note that $\triangle A'B'C'$ could be alternatively constructed as the image of $\triangle A_1B_1C_1$ via a homothety with center P and ratio 1:2, following the famous fact that every centroid divides the medians in ratio 2:1 counted from the vertices.)

It remains to show that $\triangle A'B'C'$ coincides with $\triangle ABC$. By contradiction, suppose that they are not the same triangle. Without loss of generality, we may assume that point A is different from point A'. By hypothesis and construction, both A and A' lie on line A_1P, and similarly both B and B' lie on line B_1P, and both C and C' lie on line C_1P. Since A and A' are different, it is not hard to see that there are only two possibilities for their relative positions: A' is between A and P, or A is between $A'$and P. In the first case, A is outside of $\triangle A'B'C'$, while in the second case, A is inside $\triangle A'B'C'$. If we are in the first case, drawing line AC_1 to intersect $B'B_1$ in B places B inside $\triangle A'B'C'$; then drawing line BA_1 to intersect $C'C_1$ in C places C outside $\triangle A'B'C'$. Thus, $\triangle APC$ properly contains $\triangle A'PC'$, and hence B_1 is inside $\triangle APC$; well, this certainly doesn't make sense since B_1 is supposed to lie on side AC by hypothesis. This contradiction eliminates the first case mentioned above.

The second case (A between A' and P) is treated similarly and leads to point B_1 being outside $\triangle APC$, while it is supposed to lie on side AC: again a contradiction.

This exhausts all possible situations and implies that the initial assumption of A and A' being different is false. But if $A = A'$, one quickly discovers that B and B' must also coincide, as well as C and C'. In conclusion, $\triangle ABC$ is $\triangle A'B'C'$ constructed earlier, and hence $\triangle ABC$'s centroid is already shown to be point P. This concludes the proof. ∎

Solution 2: The first part of the problem: "if P is the centroid of $\triangle ABC$ then P is also the centroid of $\triangle A_1B_1C_1$", is shown the way same as in Solution 1.

For the second part, suppose that P is the centroid of $\triangle A_1B_1C_1$. Denote by A_2, B_2 and C_2 the midpoints of the corresponding sides of $\triangle A_1B_1C_1$ so that A, A_2, P, A_1 are collinear, etc. We shall use the following:

[Part of] Menelaus Theorem. Given $\triangle XYZ$, let line l intersect line XY in point L, line YZ in point M, and line ZX in point N. (L, M and N can each be inside or outside of the corresponding side of $\triangle XYZ$.) Then

$$\frac{XL}{LY} \cdot \frac{YM}{MZ} \cdot \frac{ZN}{NX} = 1.$$
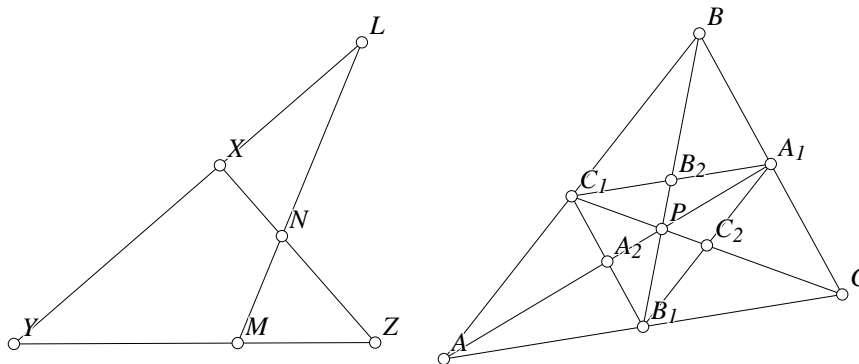
Apply Menelaus Theorem to $\triangle B_1C_1B$ and line AA_2P:

$$\frac{C_1A}{AB} \cdot \frac{BP}{PB_1} \cdot \frac{B_1A_2}{A_2C_1} = 1.$$

Since $B_1A_2 = A_2C_1$, we obtain

$$\frac{C_1A}{AB} = \frac{PB_1}{BP}.$$ (1)

Using now $\triangle B_1A_1B$ and line CC_2P, we similarly obtain

$$\frac{A_1C}{CB} = \frac{PB_1}{BP}.$$ (2)

Equating (1) and (2) yields:

$$\frac{C_1A}{AB} = \frac{A_1C}{CB} \cdot \Rightarrow 1 - \frac{C_1A}{AB} = 1 - \frac{A_1C}{CB} \Rightarrow \frac{AB - C_1A}{AB} = \frac{CB - A_1C}{CB} \Rightarrow \frac{C_1B}{AB} = \frac{A_1B}{CB}.$$

The last equality, and the fact that $\triangle C_1BA_1$ and $\triangle ABC$ share the same angle $\angle B$, shows that these two triangles are similar, from which we can conclude that C_1A_1 is parallel to AC. Applying similar arguments, one shows that A_1B_1 is parallel to BA, and B_1C_1 is parallel to CB. Now we are in a situation identical to the construction of $\triangle A'B'C'$ in Solution 1, where we easily showed that P is the centroid of $\triangle A'B'C'$, and hence in our present situation, P is the centroid of $\triangle ABC$. ■

4 Suppose that n squares of an infinite square grid are colored grey, and the rest are colored white. At each step, a new grid of squares is obtained based on the previous one, as follows. For each location in the grid, examine that square, the square immediately above, and the square immediately to the right. If there are two or three grey squares among these three, then in the next grid, color that location grey; otherwise, color it white. Prove that after at most n steps all the squares in the grid will be white.

Below is an example with $n = 4$. The first grid shows the initial configuration, and the second grid shows the configuration after one step.
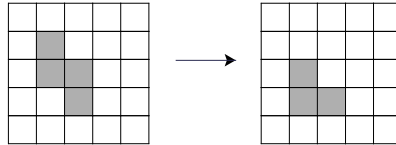
Solution: (Sketch) Use strong induction. Consider the smallest rectangle R that contains all the black squares. Suppose this rectangle contains r squares. Assume any rectangle that contains $k < r$ squares will convert to all white squares after k steps. Since R is the smallest rectangle, its left-most column and its bottom row must contain some black squares. So the rectangle with the left-most column of R removed and the rectangle with the bottom row removed must both have less than r black squares. So both of those rectangles must be converted to all white after $r - 1$ steps. (We are using the fact that squares below and to the left of a rectangle cannot affect the evolution of the squares within that rectangle.) So, after $r - 1$ steps, R would have been converted to all white squares except for possibly the lower left corner square. But even if that square is black, the rth conversion will convert that square into white square. ■

5 We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

Solution: Number the switches $1, 2, \ldots, k$. For any given configuration of switches, let the "height" of the configuration be the product of all values of n for which switches $n - 1$ and n point in the same direction (or 1 if there are no such n); this is always a positive integer. We claim that the height increases with every operation. Indeed, consider the operation in which switches $n, n+1, n+2$ are turned. Before this operation, switches $n - 1$ and n may (or may not) have pointed in the same direction, as may $n + 2$ and $n + 3$; no other such pairs can be broken, so the height is divided by at most $n(n + 3)$. However, the pairs $n, n+1$ and $n+1, n+2$ are created, multiplying the height by $(n+1)(n+2)$. Thus if h is the height of a configuration, then the new height h' after one move satisfies:

$$h' \geq h \cdot \frac{(n+1)(n+2)}{n(n+3)} = h \cdot \frac{n^2 + 3n + 2}{n^2 + 3n} > h.$$

So, as claimed, the height does increase at every step. Since the height is an increasing integer, and it cannot exceed $1 \times 2 \times \cdots \times k$, it can only increase finitely many times, and the result follows. ■

REMARK: There are other height functions. For example, if h denotes the sum of all numbers $\sqrt{k}$ for which switches k and $k + 1$ point in different directions, then it can be easily proved (as in the solution above) that h can only decrease. Since there are only finitely many values for h, one cannot perform the moves infinitely long.

More advanced readers can attempt to construct more height functions using "concave-down" functions. The goal is to come up with a positive valued function $f(n)$ for all positive integers n such that, for example $f(n) \cdot f(n+3) < f(n+1) \cdot f(n+2)$ for all such n, or $f(n) + f(n+3) < f(n+1) + f(n+2)$ for all such n.