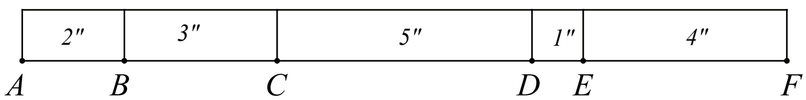
9th Bay Area Mathematical Olympiad

February 27, 2007

Problems with Solutions

1 A 15-inch-long stick has four marks on it, dividing it into five segments of length $1, 2, 3, 4$, and 5 inches (although not neccessarily in that order) to make a "ruler." Here is an example.



Using this ruler, you could measure 8 inches (between the marks B and D) and 11 inches (between the end of the ruler at A and the mark at E), but there's no way you could measure 12 inches.

Prove that it is impossible to place the four marks on the stick such that the five segments have length $1, 2, 3, 4$, and 5 inches, and such that every integer distance from 1 inch through 15 inches could be measured.

Solution 1: In order to measure 14 inches, one mark must be 1 inch from an end of the ruler. Likewise, in order to measure 13 inches, there must be another mark that is 2 inches from an end of the ruler. Without loss of generality, suppose the leftmost mark is 1 inch from the end, and the rightmost mark is 2 inches from the other end.

Next, we observe that the second mark from the left must be 5 inches from the first, or else it would be impossible to measure 6 inches.

At this point, there are only two cases to consider: either the distances between marks are, in order,

$$1, 5, 4, 3, 2;$$

or the distances are

$$1, 5, 3, 4, 2.$$

In the first case, we cannot measure 8 inches, and in the second case, we cannot measure 10 inches. We conclude that it is impossible. ∎

Solution 2: In order to make a measurement, you must choose a pair of marks, where the two endpoints are included. Since there are 6 marks, that means that we can measure at most 15 different segments ($6 \times 5/2$). Therefore a ruler that can measure every length from 1" through 15" must have exactly one way of measuring each length.

If the 1" segment is next to the 2", 3", or 4" segments, then that would make a second way of measuring 3", 4", or 5". Therefore, the 1" segment must be next to the 5" segment only: the 1" is on the end, next to the 5". Now, if the 2" segment is next to the 3" or 4" segments, then that would make a second way of measuring 5" or 6". So the 2" segment is also next to the 5" segment only. But that leaves nowhere to put the 3” and 4” segments. ■

Solution 3: Let a,b,c,d,e be the segments in order (equalling $1,2,3,4,5$, but not necessarily in that order). As above, there are only 15 possible segments that can be measured. The sum of all 15 segments (in inches) is

$$1+2+\cdots+15 = 15\times 16/2.$$

But this is also equal to $5a+8b+9c+8d+5e$.

Hence

$$5a+8b+9c+8d+5e = 15\times 16/2 = 15\times 8.$$

Now note that $15 = a+b+c+d+e$, so

$$5a+8b+9c+8d+5e = 8a+8b+8c+8d+8e.$$

Subtract $5a+8b+8c+8d+5e$ from both sides to get $c = 3a+3e$.

But $a+e$ is at least 3 and c is at most 5, a contradiction. ■

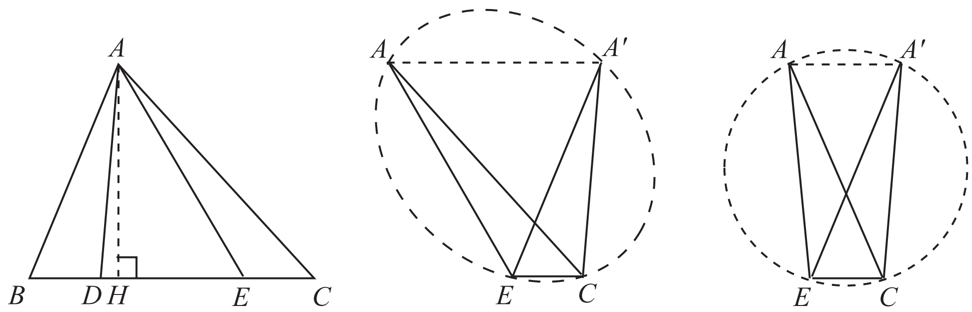
2 The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

Solution: Suppose not. Let A be a white point and B a black point. Their midpoint C is one of the two colors; without loss of generality suppose C is black. Now pick any point D not collinear with A,B,C, and construct E so that $CADE$ is a parallelogram. If D,E are both white, then $CADE$ has three white vertices and one black vertex, impossible; if they are both black, then $CADE$ has three black vertices and one white vertex, impossible. So D and E are opposite colors.

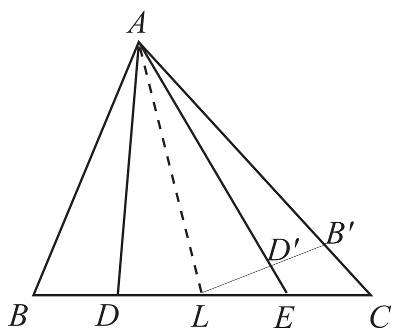
But $BCDE$ is also a parallelogram, since $BC = AC = DE$ and lines BC,DE are parallel. However, $BCDE$ it has three black vertices and one white vertex. Thus we have a contradiction. ■

REMARK: Many students wrote a flawed solution that went something like this: If there are three non-collinear black points, we can build a black parallelogram. Then we can build a new one at a different angle, etc. and thus “sweep” the entire plane with black points. This idea is good, but it doesn’t quite work, because the plane contains *uncountably* many points. It is always possible to put white points between black points. The idea of building up parallelograms gives one infinitely many black points, but infinity is not quite enough! There are plenty of holes where white points can lurk.

3 In $\triangle ABC$, D and E are two points on segment $\overline{BC}$ such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.



Solution 1: Translate $\triangle BDA$ horizontally until its side BD coincides with side EC, and label the image of point A by A'. We now have two triangles $\triangle ECA$ and $\triangle ECA'$ which share the same base EC, have the same height (equal to the height AH of the original $\triangle ABC$), and equal angles $\angle EAC$ and $\angle EA'C$. The last implies that there is a circle k passing through points E, C, A' and A. The equal heights condition implies that BC is parallel to AA'; yet the only trapezoids inscribed in a circle are isosceles. Therefore, $ECA'A$ is isosceles with $AE = A'C$. Since the diagonals of an isosceles trapezoid are equal, $EA' = CA$. Translating back to the original $\triangle ABC$, this means that $BA = CA$ and $\triangle ABC$ is isosceles. ■

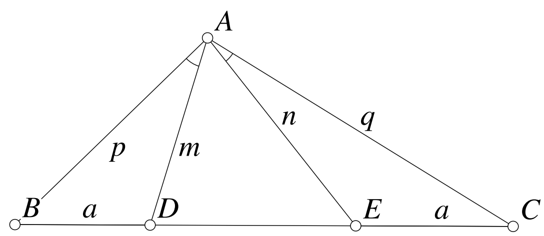


Solution 2: Without loss of generality, assume that the points B, D, E and C are arranged in this order on the line $\overleftrightarrow{BC}$; otherwise, switch points D and E in the remainder of the solution.

Suppose that $\triangle ABC$ is not isosceles. Without loss of generality, let $BA < CA$. Let AL denote the angle bisector of $\angle BAC$, where L lies on side BC. Reflect $\triangle ABL$ across AL and denote by B' and D' the images of B and D, respectively. Since $B'A = BA < CA$, B' is inside side CA, and this forces the whole segment LB' to be inside $\triangle LCA$; in particular, D' is an interior point of segment AE.

Because of the reflection, $\triangle AD'B'$ has equal area as $\triangle ADB$. In its turn, $\triangle ADB$ has the equal area as $\triangle AEC$ (they have equal bases and heights). This implies that $\triangle AD'B'$ and $\triangle AEC$ have equal area, which contradicts the fact that one triangle is properly included in the other.

We conclude that our supposition is false. Therefore, $AB = AC$ and our triangle is isosceles as desired. ■



Solution 3: Let $\angle BAD = \angle CAE = \theta$. In the end triangles, $\triangle ABD$ and $\triangle ACE$, and $\triangle ABC$, by the Law of Sines
$$\frac{a}{\sin\theta} = \frac{m}{\sin B} = \frac{n}{\sin C} \qquad \text{and} \qquad \frac{q}{\sin B} = \frac{p}{\sin C}.$$
Then $mp = nq$. Suppose $p \neq q$. Then, without loss of generality, $p < q$, which implies $np < nq = mp$ and $n < m$. But if $p < q$ then $\angle C < \angle B$ which implies $\angle AED = \angle C + \theta < \angle B + \theta = \angle ADE$ and $m < n$.
This is a contradiction, so $p = q$ and $\triangle ABC$ is isosceles. □

Solution 4: Let $\angle BAD = \angle CAE = \theta$. $\triangle ABD$ and $\triangle ACE$ have equal areas since they have equal bases ($BD = EC$) and the same altitude from A. Then $\frac{1}{2}pm\sin\theta = \frac{1}{2}qn\sin\theta$ which implies $mp = nq$. Now use the same contradiction as in the above solution. ■

4 Let N be the number of ordered pairs (x,y) of integers such that
$$x^2 + xy + y^2 \leq 2007.$$
Remember, integers may be positive, negative, or zero!

(a) Prove that N is odd.

(b) Prove that N is not divisible by 3.

Solution:

(a) If (x,y) is a pair of integers that satisfies the inequality, then $(-x,-y)$ is also such a pair, since
$$(-x)^2 + (-x)(-y) + (-y)^2 = x^2 + xy + y^2.$$
So we can match up pairs of solutions to the inequality, $(x,y) \leftrightarrow (-x,-y)$. Every solution will be paired with a different solution, except for the one remaining solution $(0,0)$ which is paired with itself. This shows that the number of solutions is odd.

(b) This is similar to the previous part, except now that we have to arrange the nonzero solutions into triples instead of pairs. If (x,y) is a solution to the inequality, then so is $(-x-y,x)$, since
$$(-x-y)^2 + (-x-y)x + x^2 = x^2 + xy + y^2.$$
Applying this transformation three times in succession gives the cycle
$$(x,y) \to (-x-y,x) \to (y,-x-y) \to (x,y),$$

so we can unambiguously arrange the solutions into cycles of three, of the form $\{(x,y),(-x-y,x),(y,-x-y)\}$. Now, if any two solutions in the same cycle are equal, then the third is also equal to them, so every cycle contains either three distinct solutions or just one solution. If a cycle contains just one solution (x,y), then $x=y$ and $y=-x-y$ gives $x=y=0$. Therefore, the solution $(0,0)$ forms a cycle by itself, and every other cycle consists of three different solutions, which means that the total number of solutions has remainder 1 when divided by 3.

NOTE: It is easier to see what is going on if we think of x and y as picking out a point in a non-rectangular coordinate system; then the number N is just the number of points of the triangular lattice inside the circle with center $(0,0)$ and radius $\sqrt{2007}$. The first transformation $(x,y)\mapsto(-x,-y)$ corresponds to a 180° rotation of the lattice; the transformation $(x,y)\mapsto(-x-y,x)$ is a 120° rotation.

5 Two sequences of positive integers, $x_1,x_2,x_3,\ldots$ and $y_1,y_2,y_3,\ldots$, are given, such that

$$y_{n+1}/x_{n+1} > y_n/x_n$$

for each $n\geq 1$. Prove that there are infinitely many values of n such that $y_n>\sqrt{n}$.

Solution: Suppose the statement is false. So there are only finitely many values for which $y_n>\sqrt{n}$; suppose there are R_1 such values. Let m be the largest integer such that $x_n/y_n\geq m$ for all n (this is possible since every $x_n/y_n>0$; note that it may be the case that $m=0$). We have $x_n/y_n<m+1$ for some n, say $n=R_2$, and since the sequence (x_n/y_n) is decreasing, we then get $x_n/y_n<m+1$ for all $n\geq R_2$. Letting $R=R_1+R_2$, we obtain

$$y_n\leq\sqrt{n} \qquad \text{and} \qquad m\leq x_n/y_n<m+1$$

for all n, with at most R possible exceptions.

Now fix any positive integer N. The pairs of positive integers (x_n,y_n) for $n<N^2$ are all distinct, since the corresponding values of y_n/x_n are strictly increasing; and except for at most R of them, the remaining ones all satisfy

$$y_n\leq\sqrt{n}<N$$

and

$$my_n\leq x_n<my_n+y_n.$$

So we have $N-1$ possible values for y_n (namely $1,2,\ldots,N-1$), and for each such value, we have y_n choices for x_n (namely $my_n,my_n+1,\ldots,my_n+y_n-1$), giving $1+2+\cdots+(N-1)=N(N-1)/2$ possible pairs obtained in this way. Hence, counting all the pairs (x_n,y_n) for $n<N^2$, we have

$$N^2-1-R\leq N(N-1)/2.$$

But since R is fixed, clearly this inequality will become false for large enough N. At this point we have a contradiction, and the problem is solved. ■

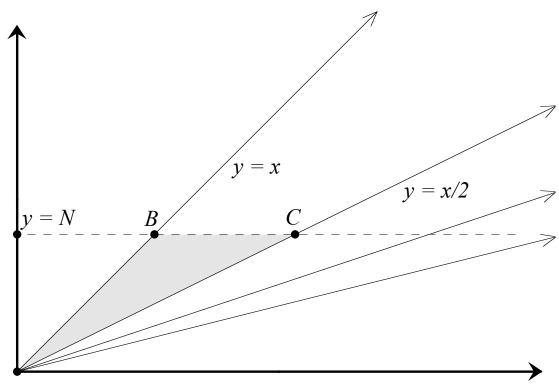
Here is an informal explanation of the above argument, using a geometric interpretation of the problem. View (x_n,y_n) as a lattice point in quadrant I of the coordinate plane, and define $m_n=$

y_n/x_n. Then m_n is the slope of the line joining (x_n, y_n) and the sequence $(x_1, y_1), (x_2, y_2), \ldots$ is a sequence of lattice points in quadrant I with *strictly increasing* slopes $m_1, m_2, \ldots$

Draw rays passing through the origin with slopes 1, $1/2, 1/3, 1/4, 1/5 \ldots$. In the picture below, we show the first four of this infinite collection of rays. These rays partition quadrant I into infinitely many "wedges." Since m_n is an increasing sequence, there will be only finitely many lattice points in all of the wedges but one, and in that "final" wedge, infinitely many lattice points will accumulate. For example, suppose that for at least one n, we have $m_n > 1/2$, and suppose that for no n do we have $m_n > 1$. Then there will be infinitely many lattice points in the wedge bounded by $y = x$ and $y = x/2$, finitely many lattice points in the wedges to the right, and no lattice points in the wedge to the left (the one bounded by the y-axis and $y = x$).

To keep things concrete, let's suppose that there are only 200 points to the right of this final wedge. Next, suppose further that there were only finitely many n such that $y_n > \sqrt{n}$. For example, suppose the largest such n was $n = 1000$. These 1200 points are "exceptional," in that 200 of them are "too far to the right" and 1000 of them are "too high." All of the rest of the points are well-behaved (they lie in the final wedge and $y_k \leq \sqrt{k}$).

Pick a large integer N (larger than 1200). Consider the triangular region in the final wedge that lies below the line $y = N$ (shaded in the figure below).



Now look at the first $N^2 - 1$ lattice points in the sequence. Of these points, at most 1200 are exceptional, and the rest of the points must lie in the shaded triangular region (since if (x_k, y_k) is a point, then $k < N^2$ and $y_k < \sqrt{k} < N$ if the point is not exceptional).

Now we can produce a contradiction: the triangular region has height N and base $BC = N$, and hence area $N^2/2$. Without worrying about boundary points, it is clear that this region cannot contain more than $N^2/2$ lattice points. However, all but 1200 of the first $N^2 - 1$ lattice points in the sequence must lie in this region. We can choose N to be arbitrarily large, but the value of 1200 will not change. So eventually, we can pick a large enough N so that there is simply no room to place the unexceptional lattice points, and we are done.

Notice that it doesn't matter which wedge is the final wedge. The area will always be $N^2/2$. There were three crucial ideas here: thinking about slopes, restricting the lattice points to a single wedge, and observing that the area of a triangle ($N^2/2$) is simply too small to put a square's worth of lattice points ($N^2 - 1200$) into!