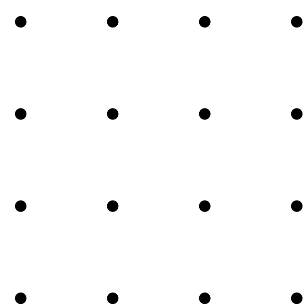
11th Bay Area Mathematical Olympiad

February 24, 2009

Problems (with Solutions)

1 A square grid of 16 dots (see the figure) contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?

Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.



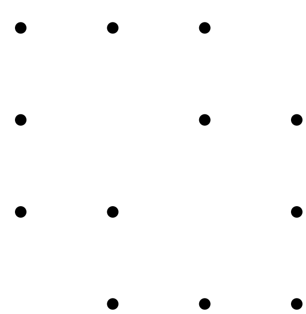
Solution:

The answer is four dots.

Four is necessary, because the four corner 1×1 squares do not share any dots in common. Other arguments that four is necessary include using the 2×2 squares, or carefully counting the number of squares eliminated by each dot (5 in the center and 3 in the corner), and since one dot must be a corner dot, then at most $3 + 5 + 5$ squares can be removed by three dots.

Removing two opposite corners of the grid and two center dots along the other diagonal provides an example to show that four is sufficient.

See the figure for one of two such possible solutions.



2 The *Fibonacci sequence* is the list of numbers that begins 1, 2, 3, 5, 8, 13 and continues with each subsequent number being the sum of the previous two.

Prove that when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example,

$$1-2+3-5=-3,$$

and 3 is an element of the Fibonacci sequence.

Solution: Expanding each number as the sum of the two previous numbers gives, for example,

$$1-2+3-5+8-13=1-(1+1)+(1+2)-(2+3)+(3+5)-(5+8).$$

Now, after removing parentheses, this is a telescoping series: each term subtracts out with its neighbor, so what's left is the last term of this expanded sum, or the second-to-last term of the original sum.

$$=1-1-1+1+2-2-3+3+5-5-8=-8.$$

To make it clear that this pattern always works, it would be nice to write all of this in terms of F_n, the n^{th} number in the list. When doing so, the previous expression becomes

$$F_1-F_2+F_3-\ldots\pm F_n=(F_1)-(F_0+F_1)+(F_1+F_2)-\ldots\pm(F_{n-2}+F_{n-1})$$

and so this works for all n because $F_0 = F_1$, so every pair of consecutive terms in this long expression adds up to 0, except for the F_{n-1} at the end.

Sketch of alternate solution:

Pair up the terms. For our example above,

$$1-2+3-5+8-13=(1-2)+(3-5)+(8-13)=1+2+5,$$

which shows that after the initial 1 we get every other term of the sequence. Then, we add, with 1+2 = 3, and then 3+5 = 8. To make it clear that this pattern always works, you will need to consider examples with an odd number of terms as well as showing that $F_1+F_2+F_4+\ldots+F_{2n} = F_{2n+1}$.

Other good arguments include a proof by mathematical induction, which supposes that this pattern works for the sum up to F_n and then proves that it continues working for the sum up to F_{n+1}.

3 There are many sets of two different positive integers a and b, both less than 50, such that a^2 and b^2 end in the same last two digits. For example, $35^2 = 1225$ and $45^2 = 2025$ both end in 25. What are all possible values for the average of a and b?

For the purposes of this problem, single-digit squares are considered to have a leading zero, so for example we consider 2^2 to end with the digits 04, not 4.

Solution: Assume that b is the larger of the two numbers. Then b^2 and a^2 end in the same last two digits, so $b^2 - a^2$ is a multiple of 100. That is, using the difference of squares to factor, $(b+a)\cdot(b-a)$ is a multiple of 100, and hence also a multiple of both 4 and 25.

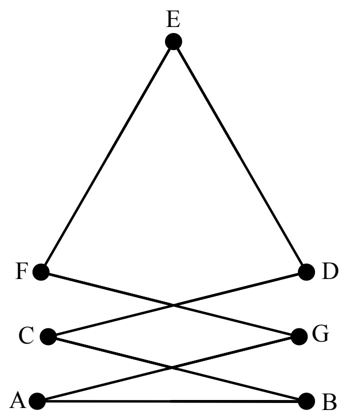
For the product to be a multiple of 25, either at least one of the numbers is a multiple of 25, or both are multiples of 5. If they are both multiples of 5, then a must be a multiple of 5, so a^2 ends in 25 or 00. There are many possibilities of this kind: Ending in 25, we have 5 and 15, 5 and 25, 5 and 35, and so on, up through 35 and 45, giving averages of 10, 15, 20, 25, 30, 35, and 40. Ending in 00, we have 10 and 20, and so on, up through 30 and 40, which gives the same list of averages. (We did give credit if you included 50 and thus put 45 in your list of possible averages.)

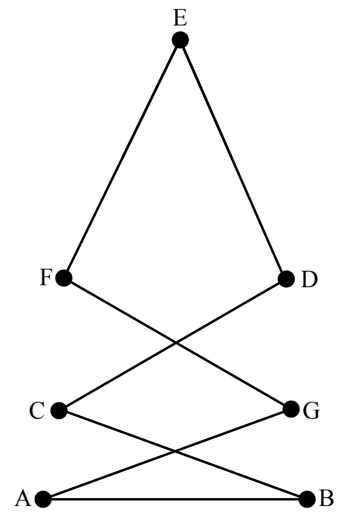
Now let's consider the possibility that one of the numbers is a multiple of 25. Neither $a-b$ nor $a+b$ could equal 25 or 75, because if they did, then one of a and b is odd, while the other is even, so their squares can't end in the same digit. Also, $b-a$ can't be 50, because $b < 50$. So the only way to obtain a multiple of 25 is for $b+a = 50$, in which case the average is 25.

Of course, you could also simply list all the squares from 1^2 through 49^2 and find all the pairs with the same last two digits!

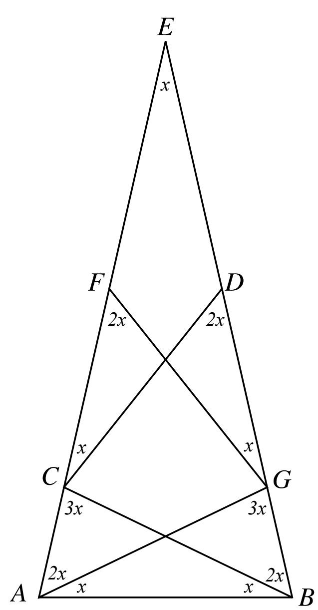
4 Seven congruent line segments are connected together at their endpoints as shown in the figure below at the left. By raising point E the linkage can be made taller, as shown in the figure below and to the right. Continuing to raise E in this manner, it is possible to use the linkage to make A, C, F, and E collinear, while simultaneously making B, G, D, and E collinear, thereby constructing a new triangle ABE.

Prove that a regular polygon with center E can be formed from a number of copies of this new triangle ABE, joined together at point E, and without overlapping interiors. Also find the number of sides of this polygon and justify your answer.





Solution: We are given $ED = EF = FG = DC = BC = AG = AB$. Let $\angle E = x$. (To aid in reading the proof, note the measures of the marked angles in the figure below.) Since $\triangle EDC$ and $\triangle EFG$ are isosceles, $\angle ECD = x = \angle EGF$. An exterior angle of a triangle has a measure equal to the sum of the remote interior angles, so $\angle AFG = 2x = \angle BDC$. Since $\triangle AFG$ and $\triangle BDC$ are isosceles, $\angle FAG = 2x = \angle DBC$. $\angle ACB$ and $\angle BGA$ are exterior angles of $\triangle ECB$ and $\triangle EGA$, respectively, so they each have measure $x + 2x = 3x$. But $\triangle ABC$ and $\triangle BAG$ are isosceles, so $\angle GBA = 3x$ and similarly $\angle CAB = 3x$. Since the base angles of $\triangle AEB$ are equal we have $AE = BE$. (Note that proving triangle ABE is isosceles was necessary for a solution to be regarded as correct.) Summing the angles of $\triangle AEB$ we get $3x + 3x + x = 7x = 180°$. Therefore $14x = 360°$ and the polygon formed by **14** such triangles is a 14-gon.



5 A set S of positive integers is called *magic* if for any two distinct members of S, i and j,

$$\frac{i+j}{GCD(i,j)}$$

is also a member of S. The GCD, or greatest common divisor, of two positive integers is the largest integer that divides evenly into both of them; for example, $GCD(36,80) = 4$.

Find and describe all finite magic sets.

Solution: Suppose there are two members i and j whose GCD is 1. Then $i+j$ is also in S. But $GCD(j,i+j)$ is also 1, so $i+2j$ is also in S. Continuing, $i+kj$ is in S for all integers k, so the set is infinite.

The only remaining possibility for a finite magic set with at least two elements is that the GCD of any pair of numbers in the set is greater than 1. Now let i and j be the two *smallest* members of S, with $i < j$. Then $\frac{i+j}{GCD(i,j)}$ is also in the set, but since the GCD is at least 2, this number is smaller than j, and thus must be equal to i.

Therefore $i+j = i \cdot GCD(i,j)$, and since the right hand side is a multiple of i, so is the left side. Thus j is a multiple of i, and thus $GCD(i,j) = i$, and so $i+j = i^2$.

Now if S has more than two elements, let n be the smallest element greater than j. By the same argument as before, $\frac{i+n}{GCD(i,n)}$ is smaller than n, but since $n > j = i^2 - i$, $n+i > i^2$, and $GCD(i,n) \leq i$, then $\frac{i+n}{GCD(i,n)} > i$. Any element of S that is between i and n must equal j, so $\frac{i+n}{GCD(i,n)} = j$, or in other words $n = (i^2-i)GCD(i,n) - i$.

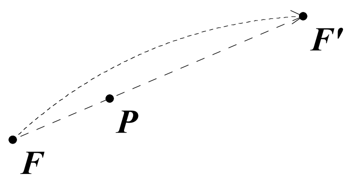
Since n is divisible by i, $GCD(i,n) = i$ and substituting gives $n = i^3 - i^2 - i$. Now consider the third pair, with j and n. Their GCD is also i, because $j = i(i-1)$ and $n = i(i^2-i-1)$, and $i^2-i-1 = (i-1)i-1$ is relatively prime to $i-1$.

Finally, $\frac{j+n}{GCD(j,n)} = \frac{(i^2-i)+(i^3-i^2-i)}{i} = i^2 - 2 < n$, so $i^2-2 = i$ or $i^2-2 = j = i^2-i$. Both cases imply that $i = 2$, from where $j = 2 = n$, a contradiction of our assumption that S has at least two elements.

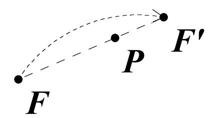
Therefore, all finite magic sets consist either of one element, or of two elements: $S = \{i, i^2 - i\}$ for any $i \geq 3$.

6 At the start of this problem, six frogs are sitting with one at each of the six vertices of a regular hexagon. Every minute, we choose a frog to jump over another frog using one of the two rules illustrated below. If a frog at point F jumps over a frog at point P, the frog will land at point F' such that F, P, and F' are collinear and:

- using Rule 1, $F'P = 2FP$.
- using Rule 2, $F'P = FP/2$.



Rule 1



Rule 2

It is up to us to choose which frog to take the leap and which frog to jump over.

(a) If we only use Rule 1, is it possible for some frog to land at the center of the original hexagon after a finite amount of time?

(b) If both Rule 1 and Rule 2 are allowed (freely choosing which rule to use, which frog to jump, and which frog it jumps over), is it possible for some frog to land at the center of the original hexagon after a finite amount of time?

Solution:

(a) Assign coordinate axes making a 120° angle so that the center of the hexagon is at $(0,0)$, the rightmost frog is at $(1,0)$, and the top left frog is at $(0,1)$. Then, the remaining four frogs are at $(1,1)$, $(-1,0)$, $(0,-1)$, and $(-1,-1)$. At each jump, if two frogs' coordinates differ by (x,y), then the jumping frog moves $(3x,3y)$. That is, each coordinate changes by a multiple of three. However, the goal has both coordinates divisible by three, and none of the frogs start with both coordinates divisible by three, so it cannot be done.

(b) One solution method is to repeat the coordinate method of the previous problem, but now encountering fractions when we use Rule 2. Since the jumping frog now moves $(3x/2, 3y/2)$ when the frogs are separated by (x,y), looking at the first coordinate a frog's jump will now be

$$\frac{p}{q} \to \frac{p}{q} + \frac{3x}{2} = \frac{2p+3qx}{2q}.$$

If p is not divisible by 3 before this jump, the numerator of this new fraction is still not divisible by 3. Since the frogs start with at least one coordinate's numerator not divisible by 3, the frogs can never reach a location where both coordinates have numerators that are divisible by 3.

Alternatively, each time Rule 2 is used, simply double all the coordinates before making the jump. This cannot change whether a frog can reach the origin, and it ensures that the coordinates remain integers and that the jumps are all by a multiple of 3, so the same argument for Rule 1 still works.

7 Let $\triangle ABC$ be an acute triangle with angles α, β, and γ. Prove that

$$\frac{\cos\alpha}{\cos(\beta-\gamma)}+\frac{\cos\beta}{\cos(\gamma-\alpha)}+\frac{\cos\gamma}{\cos(\alpha-\beta)}\geq\frac{3}{2}.$$

Solution:

Since $\cos\gamma=-\cos(\alpha+\beta)=-\cos\alpha\cos\beta+\sin\alpha\sin\beta$, and $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ the given inequality transforms to

$$\begin{aligned}\frac{\sin\beta\sin\gamma-\cos\beta\cos\gamma}{\sin\beta\sin\gamma+\cos\beta\cos\gamma}+\frac{\sin\gamma\sin\alpha-\cos\gamma\cos\alpha}{\sin\gamma\sin\alpha+\cos\gamma\cos\alpha}&\\ +\frac{\sin\alpha\sin\beta-\cos\alpha\cos\beta}{\sin\alpha\sin\beta+\cos\alpha\cos\beta}&\geq\frac{3}{2}\\ \frac{\tan\beta\tan\gamma-1}{\tan\beta\tan\gamma+1}+\frac{\tan\gamma\tan\alpha-1}{\tan\gamma\tan\alpha+1}+\frac{\tan\alpha\tan\beta-1}{\tan\alpha\tan\beta+1}&\geq\frac{3}{2}.\end{aligned}$$

Some elementary trigonometry yields

$$\begin{aligned}\tan\alpha+\tan\beta+\tan\gamma&=\frac{\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\alpha\cos\gamma+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\ &=\frac{\sin\alpha\cos\beta\cos\gamma+\cos\alpha(\sin\beta\cos\gamma+\sin\gamma\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}\\ &=\frac{\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin(\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}\end{aligned}$$

Now we use the fact that the angles add to 180° to see that $\sin(\beta+\gamma)=\sin(\alpha)$, so

$$\begin{aligned}\frac{\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin(\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}&=\frac{\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\alpha}{\cos\alpha\cos\beta\cos\gamma}\\ &=\frac{\sin\alpha\cdot(\cos\beta\cos\gamma+\cos\alpha)}{\cos\alpha\cos\beta\cos\gamma}\\ &=\frac{\sin\alpha\cdot(\cos\beta\cos\gamma-\cos(\beta+\gamma))}{\cos\alpha\cos\beta\cos\gamma}\\ &=\frac{\sin\alpha\cdot(\cos\beta\cos\gamma-(\cos\beta\cos\gamma-\sin\beta\sin\gamma))}{\cos\alpha\cos\beta\cos\gamma}\\ &=\tan\alpha\tan\beta\tan\gamma.\end{aligned}$$

Now, setting $x=\tan\alpha$ and so on, our inequality

$$\frac{yz-1}{yz+1}+\frac{zx-1}{zx+1}+\frac{xy-1}{xy+1}\geq\frac{3}{2}$$

has the property that x, y, z are positive real numbers satisfying $x+y+z=xyz$.

Multiplying the numerator and the denominator of the first fraction by x, second by y and third by z the inequality becomes

$$\frac{S-x}{S+x}+\frac{S-y}{S+y}+\frac{S-z}{S+z}\geq\frac{3}{2}, \tag{1}$$

for $S=xyz$. Using the substitution $S+x=a$, $S+y=b$, $S+z=c$ and solving for S, x, y, z we get $S=\frac{a+b+c}{4}$, $x=\frac{3a-b-c}{4}$, $y=\frac{3b-a-c}{4}$, $z=\frac{3c-a-b}{4}$, $S-x=\frac{b+c-a}{2}$, $S-y=\frac{c+a-b}{2}$, $S-z=\frac{a+b-c}{2}$, and (1) becomes

$$\frac{1}{2}\left(\frac{b}{a}+\frac{c}{a}-1+\frac{a}{b}+\frac{c}{b}-1+\frac{a}{c}+\frac{b}{c}-1\right)\geq\frac{3}{2}.$$

This immediately follows from $\frac{b}{a}+\frac{a}{b}\geq 2$, $\frac{a}{c}+\frac{c}{a}\geq 2$, and $\frac{c}{b}+\frac{b}{c}\geq 2$.

Remark. Once the inequality is transformed to (1) we can use the convexity of the function $f(t)=\frac{S-t}{S+t}$ and Jensen's inequality:

$$\frac{1}{3}\left(f(x)+f(y)+f(z)\right)\geq f\left(\frac{x+y+z}{3}\right).$$

It can be easily seen that the previous inequality is exactly (1).

Solution 2, earning a Brilliancy award for Amol Aggarwal of Saratoga HS:

Let O be the circumcenter of $\triangle ABC$. Extend AO to meet BC at X, and similarly BO extends to Y and CO to Z.

As shown in the diagram at left below, draw AA' parallel to BC, drop the altitude AH from point A, and also draw MO, the perpendicular bisector of BC, meeting AA' at N.

Now $AHMN$ is a rectangle, so $AH = MN$. Also, $ABCA'$ is a cyclic trapezoid and thus isosceles, so $\angle A'BC = \angle ACB = \gamma$. Thus $\angle ABA' = \angle ABC - \angle ACB = \beta - \gamma$, so the measure of arc $AA' = 2(\beta - \gamma)$, which means that $\angle AON = \beta - \gamma$. Using right triangle ONA, we have $ON = OA\cos(\beta - \gamma)$.

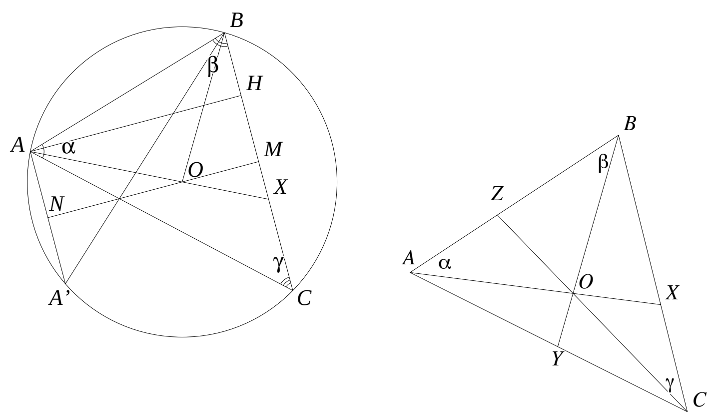
Again using central and inscribed angles, arc $BC = 2\alpha$ so $\angle BOM = \alpha$ and $OM = OB\cos\alpha$. Dividing OM by ON,

$$\frac{OM}{ON} = \frac{OB\cos\alpha}{OA\cos(\beta-\gamma)} = \frac{\cos\alpha}{\cos(\beta-\gamma)}.$$

Finally, because $AA' \parallel BC$, $\triangle ANO \sim \triangle XMO$, which gives $\frac{OX}{OA} = \frac{OM}{ON}$.

Now by symmetry, we can find similar relationships for all three fractions in the original inequality, so our goal now is to prove that

$$\frac{OX}{OA} + \frac{OY}{OB} + \frac{OZ}{OC} \geq \frac{3}{2}.$$



In the figure at right above, the area ratio $\frac{AO}{AX} = \frac{[ABOC]}{[ABC]}$, so adding analogous terms for each of the three ratios,

$$\frac{AO}{AX} + \frac{BO}{BY} + \frac{CO}{CZ} = \frac{[ABOC]+[BCOA]+[AOBC]}{[ABC]} = 2,$$

since the numerator counts the area of each piece exactly twice.

Next, by the arithmetic mean - harmonic mean inequality (or Cauchy-Schwarz if you prefer),

$$\left(\frac{AO}{AX} + \frac{BO}{BY} + \frac{CO}{CZ}\right)\left(\frac{AX}{AO} + \frac{BY}{BO} + \frac{CZ}{CO}\right) \geq 9.$$

Finally, since $\frac{AX}{AO} = 1 + \frac{OX}{AO}$ and similarly for the other two fractions, we can first divide by 2 and then subtract 3 from each side,

$$\frac{OX}{OA} + \frac{OY}{OB} + \frac{OZ}{OC} \geq \frac{9}{2} - 3 = \frac{3}{2}.$$