13th Bay Area Mathematical Olympiad

February 22, 2011

Problems (with Solutions)

1 A set of identical square tiles with side length 1 is placed on a (very large) floor. Every tile after the first shares an entire edge with at least one tile that has already been placed.

- What is the largest possible perimeter for a figure made of 10 tiles?
- What is the smallest possible perimeter for a figure made of 10 tiles?
- What is the largest possible perimeter for a figure made of 2011 tiles?
- What is the smallest possible perimeter for a figure made of 2011 tiles?

Prove that your answers are correct.

Solution: The initial single tile has perimeter 4. Each time a tile is added, at most three edges are added and at least one edge of the previous boundary is eliminated, so the perimeter increases by at most 2. Thus for n tiles the maximum possible perimeter is $2(n+1)$ and this can be achieved (among other ways) by placing all of the tiles in a straight line. Therefore, the maximum perimeter for 10 tiles is 22 and for 2011 tiles, 4024.

The perimeter of a pattern is always greater than or equal to the perimeter of the smallest rectangle that encloses that pattern. To see this, project each side of the enclosing rectangle toward the pattern and every unit length of the rectangle will land on a different edge of the pattern. Any extra edges of the pattern, including concave "dents" and internal edges that surround holes, will only add to the pattern's perimeter.

Let s and t be the side lengths of a rectangle enclosing the pattern. Then, since the pattern fits inside the rectangle, $st \geq n$. Also, from the previous paragraph, the perimeter $p \geq 2(s+t)$, so we have

$$\begin{aligned} p^2 &\geq 4(s+t)^2 \\ &\geq 4((s+t)^2-(s-t)^2) \quad \text{since } (s-t)^2 \text{ is not negative} \\ &= 4(4st) \\ &\geq 16n \end{aligned}$$

The inequality $p \geq 4\sqrt{n}$ follows by taking the square root of both sides.

Since each tile has an even number of sides and sides are shared in pairs, the perimeter must be an even number.

For $n = 10$, $4\sqrt{10} > 12$, so the perimeter is at least 14. The number 14 can be achieved, for example, by placing the 10 tiles in a 2×5 rectangle, or in several patterns that fit inside a 3×4 rectangle.

For $n = 2011$, $4\sqrt{2011} > 178$, so the perimeter is at least 180. The number 180 can be achieved, for example, by filling a 45×45 rectangle and chopping off a line of $2025 - 2011 = 14$ tiles along an edge, beginning at a corner. (Several non-square starting rectangles will also work.)

To generalize this result, the minimal perimeter can always be obtained by fitting the pattern inside the smallest possible $n \times n$ or $n \times n-1$ rectangle, and sometimes in other ways as well.

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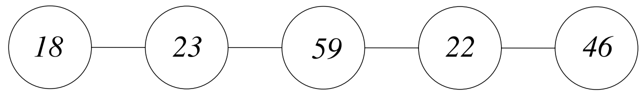
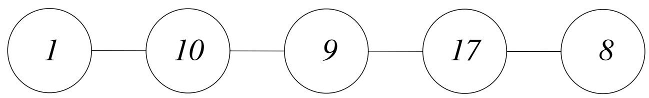
2 Five circles in a row are each labeled with a positive integer. As shown in the diagram, each circle is connected to its adjacent neighbor(s). The integers must be chosen such that the sum of the digits of the neighbor(s) of a given circle is equal to the number labeling that point. In the example, the second number $23 = (1+8)+(5+9)$, but the other four numbers do not have the needed value.



What is the smallest possible sum of the five numbers? How many possible arrangements of the five numbers have this sum? Justify your answers.

Solution: The smallest sum is **45**, and one example arrangement is shown in the diagram.



If the leftmost number has two or more digits, it is at least 10, so the next number must be at least 19 to have a big enough digit sum. Then the next number would have to be a three-digit number, so the sum would be much larger than 45.

Thus, we only need to consider the case where the leftmost number is a single digit a. The next number must be greater than a, but its digits must sum to a, so it must be at least $10+(a-1)$, or in other words $a+9$. Therefore the middle number has a digit sum of at least 9, so the number itself must be at least 9. The fourth number is greater than the digit sum of the third number, so it must have at least two digits, and these digits must sum to at least $9-a$ in order to make the middle number equal the sum of its neighbors' digits. Thus, the fourth number is at least $10+(9-a-1)$, or in other words $18-a$. Finally the last number equals the digit sum of the fourth number, so it is at least $9-a$.

Summing the five numbers, we have $a+(a+9)+9+(18-a)+(9-a)$, which is equal to 45 for any value of a. So there are 8 solutions, one for each possible value of the single digit a, 1 through 8, which leave all five circles filled with positive numbers.

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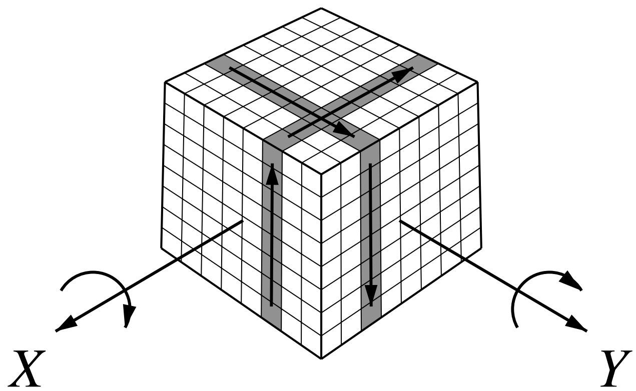
3 Consider the $8 \times 8 \times 8$ Rubik's cube below. Each face is painted with a different color, and it is possible to turn any layer, as you can with smaller Rubik's cubes. Let X denote the move that turns the shaded layer shown (indicated by arrows going from the top to the right of the cube) clockwise by 90 degrees, about the axis labeled X. When move X is performed, the only layer that moves is the shaded layer. Likewise, define move Y to be a clockwise 90-degree turn about the axis labeled Y, of just the shaded layer shown (indicated by the arrows going from the front to the top, where the front is the side pierced by the X rotation axis). Let M denote the move "perform X, then perform Y."



Imagine that the cube starts out in "solved" form (so each face has just one color), and we start doing move M repeatedly. What is the least number of repeats of M in order for the cube to be restored to its original colors?

Solution: There are two "bands" of individual unit cubes (called *cubies*) that are moved by M. Of those, only the cubies exactly three units from an edge of the cube, such as the cubie originally at the intersection of the shaded bands, can ever move out of a single plane of rotation. All the other cubes either remain fixed or return to their original position every four repetitions of M as they travel once around the cube with four 90 degree turns.

Each of these cubies that move in both the planes of rotation will return to its original position every seven repetitions of M. It must be moved by a total of four 90 degree turns around each of X and Y to return to its starting point; to do so, it moves once with both X and Y, three times with X, and three times with Y, in some order depending on the cubie's initial position. For example, the cubie that is initially at the top intersection of the two shaded bands will move first to the right face (where the Y arrow emerges) and then to the bottom and then the left. On the fourth move M it will be affected by first X and then Y, ending up on the back. Then it is affected only by Y for three moves, shifting to the bottom, front, and finally returning to its original location on top after seven moves.

Thus, for all the cubies to return to their starting points, the number of moves M must be a multiple of both 4 and 7, and therefore 28 is the smallest possible number of repeats of M in order for the cube to be restored.

■

4 In a plane, we are given line l, two points A and B neither of which lies on line l, and the reflection A_1 of point A across line l. Using only a straightedge, construct the reflection B_1 of point B across line l. Prove that your construction works.

Note: "Using only a straightedge" means that you can perform only the following operations:

(a) Given two points, you can construct the line through them.

(b) Given two intersecting lines, you can construct their intersection point.

(c) You can select (mark) points in the plane that lie on or off objects already drawn in the plane. (The only facts you can use about these points are which lines they are on or not on.)

Solution: We are given a line, a pair of distinct points A and A_1 that are reflections of each other across that line and a third point B waiting to be reflected across the line. We can assume that the point labeled A is on the same side of the line as B.

Given line l and two points A and B on the same side of the line, there are three possible ways line l and line $\overleftrightarrow{AB}$ can align on the plane:

(1) line l and line $\overleftrightarrow{AB}$ intersect at an oblique (non-right) angle

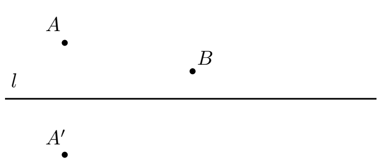
(2) line l is parallel to line $\overleftrightarrow{AB}$

(3) line l is perpendicular to line $\overleftrightarrow{AB}$

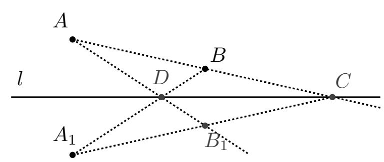
These three possibilities set up three cases. A solution to the first case creates a straightedge procedure that can be easily used to solve the other two cases.

Case 1:

There are two key ideas to this construction. First, The reflection of any point on line l is the point itself. Second, the intersection of two reflection lines is the reflection of the intersection of the two original lines. The general plan is to reflect two lines that intersect at B. We do this by having lines that go through B and contain two points whose reflections are known.



Setup



Complete diagram

- C is where line l meets $\overleftrightarrow{AB}$
- D is where line l meets $\overline{A_1B}$
- B is where $\overrightarrow{AD}$ meets $\overline{A_1C}$

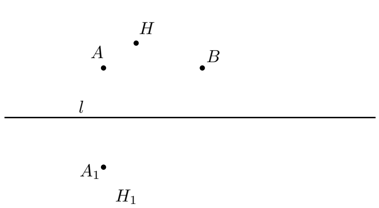
Procedure

$\overleftrightarrow{AC}$ is the first line containing B, and $\overleftrightarrow{A_1C}$ is its reflection. $\overleftrightarrow{A_1D}$ contains B and has reflection $\overleftrightarrow{AD}$. Since B is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{A_1D}$, the reflection of B is the intersection of $\overleftrightarrow{A_1C}$ and $\overleftrightarrow{AD}$.

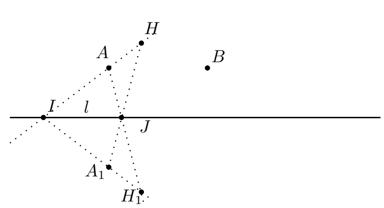
Note that we now have a straightedge procedure that allows us to reflect a point (the *target*) across a line if we have a pair of points that we use as *guide points* of a reflection across the line. This works when the target point (B) and the same side example point (A) are on a line that intersects the line of reflection (l).

Case 2:

In this case, line $\overleftrightarrow{AB}$ is parallel to line l. No point C as in case 1 is available. We can use straightedge operation (c) and select a point H that is not on line $\overleftrightarrow{AB}$ and not on line $\overleftrightarrow{AA_1}$.



Select any appropriate point H.

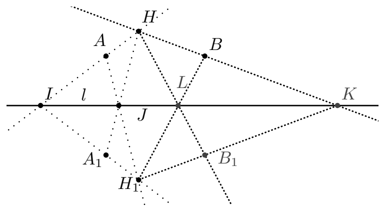


Use case 1 on H, A, A_1 to get H_1.

- select H so that $\overleftrightarrow{HA}$ is neither parallel nor perpendicular to line l
- $I =$ line $l \cap \overleftrightarrow{AH}$
- $J =$ line $l \cap \overline{A_1H}$
- $H_1 = \overrightarrow{AJ} \cap \overrightarrow{IA_1}$

Procedure to reflect H.

Now we are set to use our straightedge procedure to reflect H across line l using points A and A_1 as the guide points. This is shown in figure 5. Finally we can use H and H_1 as guide points for reflecting B across line l as shown in figure 6.



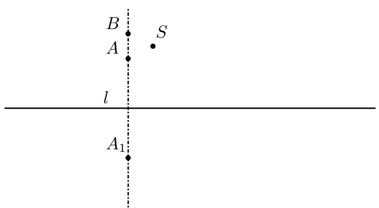
Use H and H_1 to construct B_1

- K is where line l meets $\overleftrightarrow{BH}$
- L is where line l meets $\overline{H_1B}$
- B_1 is where $\overrightarrow{HL}$ meets $\overline{H_1K}$

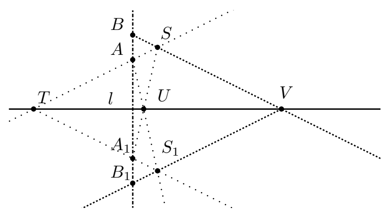
Procedure using H and H_1 to construct B_1

Case 3:

When $\overleftrightarrow{BA}$ is perpendicular to line l, B is on $\overleftrightarrow{AA_1}$ and so will be B_1. As in case 2, we need to pick a point S so that line $\overleftrightarrow{AS}$ is neither parallel nor perpendicular to line l. Again we use A and A_1 as guide points to reflect S. Now using S and S_1, we can get B_1 as shown below.



With B on line AA_1, select S.



Get S_1; use S and S_1 to construct B_1.

- Select H so that $\overleftrightarrow{HA}$ is neither parallel nor perpendicular to line l
- T is where line l meets $\overleftrightarrow{AS}$
- U is where line l meets $\overline{A_1S}$
- S_1 is where $\overrightarrow{AU}$ meets $\overrightarrow{TA_1}$

- V is where line l meets $\overleftrightarrow{BS}$
- B_1 is where $\overrightarrow{VS_1}$ meets $\overrightarrow{BA_1}$

Procedure

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5 Let S be a finite, nonempty set of real numbers such that the distance between any two distinct points in S is an element of S. In other words, $|x-y|$ is in S whenever $x \neq y$ and x and y are both in S.

Prove that the elements of S may be arranged in an arithmetic progression. This means that there are numbers a and d such that $S = \{a, a+d, a+2d, a+3d, \ldots, a+kd, \ldots\}$.

Solution: If S has just one element, then $S = \{a\}$ satisfies the given condition. Let the elements of S be $a_1 < a_2 < \cdots < a_n$. If $a_1 < 0$, then $|a_n - a_1| \geq a_n - a_1 > a_n$, so $|a_n - a_1|$ cannot be an element of S — contradiction.

Now suppose $a_1 > 0$. The numbers

$$\begin{aligned} b_2 &= a_2 - a_1 \\ b_3 &= a_3 - a_1 \\ &\vdots \\ b_n &= a_n - a_1 \end{aligned}$$

are all positive differences of members of S, so each is in S. But each b_k is less than a_k, and greater than $b_2, \ldots, b_{k-1}$, so by induction we must have $b_k = a_{k-1}$. Thus, $a_{k-1} = a_k - a_1$ for $k = 2, 3, \ldots, n$, which gives $k = ka_1$ for each k. Thus the members of S form the arithmetic progression $a_1, 2a_1, 3a_1, \ldots, na_1$.

Finally, if $a_1 = 0$, then we can remove a_1 and the remaining members of S still satisfy the condition of the problem. Then, by the previous case, they are $a_2, 2a_2, \ldots, (n-1)a_2$. Hence, the members of S form the arithmetic progression $0, a_2, 2a_2, \ldots, (n-1)a_2$.

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6 Three circles k_1, k_2, and k_3 intersect in point O. Let A, B, and C be the second intersection points (other than O) of k_2 and k_3, k_1 and k_3, and k_1 and k_2, respectively. Assume that O lies inside of the triangle ABC. Let lines AO, BO, and CO intersect circles k_1, k_2, and k_3 for a second time at points A', B', and C', respectively. If $|XY|$ denotes the length of segment XY, prove that

$$\frac{|AO|}{|AA'|}+\frac{|BO|}{|BB'|}+\frac{|CO|}{|CC'|}=1.$$

Solution: In this solution we will use a method called *Inversion in the Plane*.[1]

We invert with respect to point O with an arbitrary radius r. We will label the images of objects (points, circles, lines, segments) under this inversion by putting a bar over them. By properties of inversion, the three given circles through O will invert to three lines *not* through O. For instance, circle k_1 will invert to a line $\overline{k_1}$ through points $\overline{B},\overline{C}$, and $\overline{A'}$, and similarly for circles k_2 and k_3. On the other hand, the original line AOA' will invert to itself (as it passes through the center of inversion O); now, points A and A' will move to, perhaps different, points $\overline{A}$ and $\overline{A'}$ on this same line, while point O will stay where it is (we shall not apply inversion here to the center of inversion O). Analogous situations occur for lines BOB' and COC'.

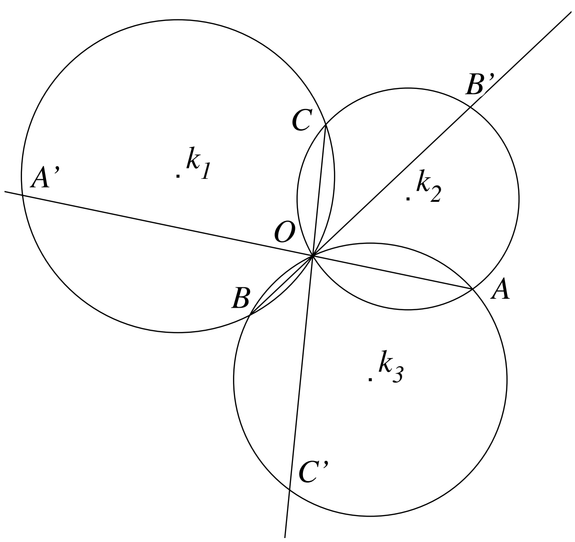


Diagram before inversion

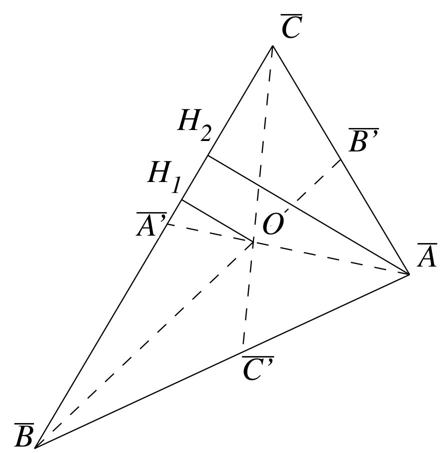


Diagram after inversion, with perpendiculars to $\overline{BC}$

We are ready to describe the new inverted picture. Point O is inside triangle $\overline{ABC}$. Three points $\overline{A'},\overline{B'}$, and $\overline{C'}$ are chosen on the triangle's sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, so that the three segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ all intersect in point O (such segments are called *cevians* in $\triangle\overline{ABC}$). Two distance formulas relating new to old distances under inversion tell us:

$$|AO|=\frac{r^2}{|O\overline{A}|},\text{ and }|AA'|=\frac{r^2\cdot|\overline{AA'}|}{|O\overline{A}|\cdot|O\overline{A'}|}.$$

Dividing these two expressions and canceling r^2 and $|O\overline{A}|$ re-expresses one of the desired ratios completely in terms of the new inverted picture:

$$\frac{|AO|}{|AA'|}=\frac{|O\overline{A'}|}{|\overline{AA'}|}.$$

Now drop perpendiculars OH_1 and $\overline{A}H_2$ to side $\overline{BC}$ as shown in the inverted picture, and call their lengths h_1 and h_2. This creates two similar triangles: $\triangle O\overline{A'}H_1 \sim \triangle\overline{AA'}H_2$: they share an angle and have another right angle each. Hence, the ratios of corresponding sides are equal:

$$\frac{|O\overline{A'}|}{|\overline{AA'}|}=\frac{|OH_1|}{|\overline{A}H_2|}=\frac{h_1}{h_2}=\frac{h_1\cdot|\overline{BC}|/2}{h_2\cdot|\overline{BC}|/2}=\frac{S_{\triangle O\overline{BC}}}{S_{\triangle\overline{ABC}}},$$

[1]There are a number of books where one can read about inversion. For example, an introduction to the method of inversion is the content of Chapter 1 in *"A Decade of the Berkeley Math Circle – The American Experience"*, edited by Zvezdelina Stankova and Tom Rike, published by AMS/MSRI.

where h_1 and h_2 are the lengths of the drawn altitudes OH_1 and $\overline{A}H_2$ in $\triangle O\overline{BC}$ and $\triangle\overline{ABC}$, respectively. Along the way, we multiplied by $|\overline{BC}|/2$ to recreate the standard formulas for the areas of $O\overline{BC}$ and $\triangle\overline{ABC}$, and denoted correspondingly those areas by $S_\triangle$ in the last ratio.

Of course, we can repeat the above discussion for the other two ratios $\frac{|BO|}{|BB'|}$ and $\frac{|CO|}{|CC'|}$, and end up rewriting the desired sum in a completely different way:

$$\frac{|AO|}{|AA'|}+\frac{|BO|}{|BB'|}+\frac{|CO|}{|CC'|}=\frac{S_{\triangle O\overline{BC}}}{S_{\triangle\overline{ABC}}}+\frac{S_{\triangle O\overline{CA}}}{S_{\triangle\overline{ABC}}}+\frac{S_{\triangle O\overline{AB}}}{S_{\triangle\overline{ABC}}}=\frac{S_{\triangle O\overline{BC}}+S_{\triangle O\overline{CA}}+S_{\triangle O\overline{AB}}}{S_{\triangle\overline{ABC}}}=\frac{S_{\triangle\overline{ABC}}}{S_{\triangle\overline{ABC}}}=1.$$

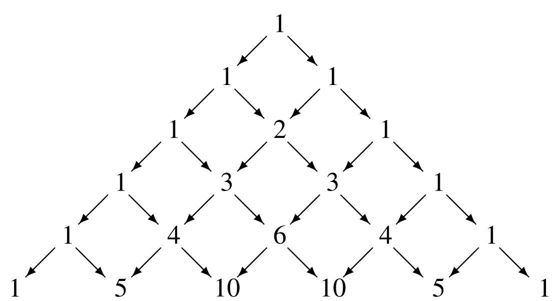
Here we used the fact that O is inside $\triangle\overline{ABC}$ so that the three triangles with vertex O, $\triangle O\overline{BC}$, $\triangle O\overline{CA}$, and $\triangle O\overline{AB}$, make up the whole big $\triangle\overline{ABC}$, and hence their areas add up to the area of this big triangle.

■

7 Does there exist a row of Pascal's Triangle containing four distinct elements a,b,c and d such that $b=2a$ and $d=2c$?

Note that the values must be distinct, so a,b,c,d must be four different numbers.

Recall that Pascal's triangle is the pattern of numbers that begins as follows



where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4+6$. Also note that the last row displayed above contains the four elements $a=5, b=10, d=10, c=5$, satisfying $b=2a$ and $d=2c$, but these four elements are NOT distinct.

Solution: Yes, there are infinitely many such rows. For example,

$$\binom{203}{68}=2\binom{203}{67} \quad \text{and} \quad \binom{203}{85}=2\binom{203}{83}.$$

There are infinitely many rows having two *adjacent* elements in a 1:2 ratio, for

$$2\binom{n}{k}=\binom{n}{k+1}$$

reduces to $2(k+1)=n-k$, or $n=3k+2$. So as long as $n\equiv 2 \pmod 3$, there will be two adjacent elements in a 1:2 ratio.

Next, we search for "doubles" that are not adjacent. The next easiest case to try is

$$2\binom{n}{k}=\binom{n}{k+2},$$

which reduces to

$$2(k+2)(k+1)=(n-k)(n-k-1).$$

Substitute $u=n-k$ and $v=k+2$; our equation becomes

$$2(v^2-v)=u^2-u.$$

Multiplying both sides by 4 and completing the square yields

$$2(4v^2-4v+1)=4u^2-4u+1+1,$$

so substituting $x=2v-1, y=2u-1$ reduces the original equation to

$$2x^2-y^2=1.$$

This is a Pell's equation with infinitely many solutions which can be generated in the standard way, or we can observe that $(1,1),(5,7)$ are solutions and that if (x,y) is a solution, then $(3x+2y,4x+3y)$ is also a solution.

It remains to show that one of these solutions produces an n which is congruent to 2 modulo 3. Since $n=\frac{x+y}{2}-1$, we must have $x+y\equiv 0 \pmod 6$. The first solution with this property is $(5,7)$, but this doesn't work, because it corresponds to $n=5$, and the elements of the row are $1,5,10,10,5,1$, so the doubles are not distinct. Generating solutions via $(x,y)\rightarrow(3x+2y,4x+3y)$ modulo 6, we have the repeating pattern

$$(1,1),(-1,1),(-1,-1),(1,-1),(1,1),\ldots,$$

which shows that every other solution (x,y) has the property $x+y\equiv 0 \pmod 6$. The next solution after $(5,7)$ with this property will correspond to an n large enough so that the doubles will be distinct. Specifically, the solution is $(169,239)$ which corresponds to $n=203, k=83$.

NOTE: There are no rows before $n=203$ which have two distinct double pairs. I know of no easy way to see this without a computer search. ■