14th Bay Area Mathematical Olympiad

BAMO Exam

February 28, 2012

Problems with Solutions

1 Hugo plays a game: he places a chess piece on the top left square of a 20×20 chessboard and makes 10 moves with it. On each of these 10 moves, he moves the piece either one square horizontally (left or right) or one square vertically (up or down). After the last move, he draws an X on the square that the piece occupies. When Hugo plays this game over and over again, what is the largest possible number of squares that could eventually be marked with an X? Prove that your answer is correct.

Solution: Index each square by its row number and column number, starting with 0. For example, (0,0) represents the top left square and (2,5) represents the square in the third row down and the sixth column over. When the piece moves down or to the right, the sum of the indices of its square increases by 1, and when the piece moves up or to the left, this sum decreases by 1. Since it starts on a square with sum of indices 0, after 10 moves it must lie on a square with sum of indices at most 10. In addition, since each move changes the sum of indices from even to odd or from odd to even and the piece starts on a square with an even sum of indices, after an even number of moves the sum of indices must be even. Therefore, after 10 moves, the piece lies on a square whose sum of indices is an even number ≤ 10.

It is possible to reach any one of the squares with sum of indices an even number ≤ 10 at the end of 10 moves, since the piece can get to the square (i, j) with $i + j \leq 10$ simply by moving i squares down, then j squares to the right. If $i + j = 10$, this uses up all 10 moves; otherwise, the piece can waste the remaining $10 - i - j$ moves (which is an even number of moves since $i + j$ is even) simply by moving the piece down a square and then up a square until 10 moves are reached.

We have shown that the squares that could be marked with an X are the squares of indices (i, j) where $i + j$ is an even number ≤ 10. Since the squares with $i + j = n$ form a diagonal of length $n + 1$ extending from the left side of the board to the top of the board, there are $1 + 3 + 5 + 7 + 9 + 11 = 36$ such squares.

Note that if the chessboard is colored black and white in the usual way with a black square in the top left corner, then these squares are the top left square together with the next 5 black diagonals that run from the left side to the top of the board. ■

2 Answer the following two questions and justify your answers:

(1) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + 5^{2012}$?

(2) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + \cdots + 2011^{2012} + 2012^{2012}$?

Solution: The final digit of a power of k depends only on the final digit of k, so there are 10 cases to consider. These are easy to work out. For k ending in 1, the final digits are $1, 1, 1, 1, \ldots$. For k ending in 2 they are $2, 4, 8, 6, 2, 4, 8, 6, \ldots$, et cetera. In fact all 10 possible final digits repeat after 1, 2 or 4 steps, so in every case the final digit is back where it started every 4 steps. Since 2012 is divisible by 4, the last digit of k^{2012} is the same as the last digit of k^4. As k varies, the last digits of k^4 go through a cycle of length 10: $1, 6, 1, 6, 5, 6, 1, 6, 1, 0$.

For part (1), if we list the last digits of the five summands, we have $1, 6, 1, 6, 5$, whose sum has a last digit of 9.

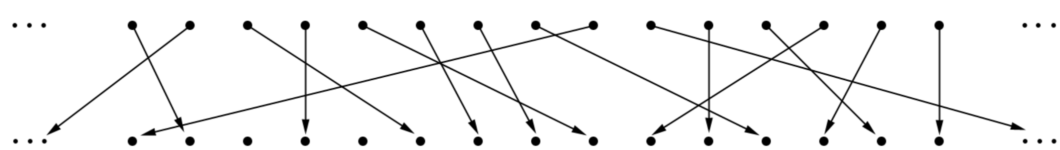
For part (2), if we list the last digits of the 2012 summands, we will have 201 copies of the sequence $1, 6, 1, 6, 5, 6, 1, 6, 1, 0$, followed by 1 and 6. Since $1+6+1+6+5+6+1+6+1+0 = 33$, the last digit of the original sum is the same as the last digit of $201 \cdot 33 + 1 + 6$, which is 0.

■

3 Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:

- No two arrows point at the same dot.
- No arrow can extend right or left by more than 1006 positions.

Show that at most 2012 dots in the lower row could have no arrow pointing to them.



Solution: Call dots in the lower line that lie at the endpoints of arrows "target dots" and those that are not, "missed dots". If an arrangment has 2013 or more missed dots, pick a contiguous set S of dots in the lower line that includes exactly 2013 missed dots and t target dots.

Consider the set of $t + 2013$ dots directly above the dots in S from which $t + 2013$ arrows must initiate. At most t of them can terminate in S, so at least 2013 of them terminate outside S. But since arrows can only extend to dots 1006 outside of S on either side, there are only $1006 + 1006 = 2012$ possible targets for those 2013 or more arrows, which is impossible.

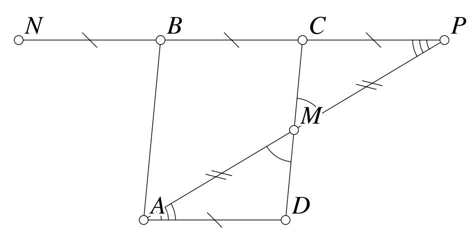
Therefore it is impossible to have 2013 or more missed dots in a valid configuration. ■

4 Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane. She can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler. With it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.

Solution: Laura should extend the line AM beyond M. Measure AM and find the point P on the extension of AM beyond M such that $AM = MP$. Vertical angles $\angle CMP = \angle DMA$, $CM = MD$ and $AM = MP$ so $\triangle PMC$ is congruent to $\triangle AMD$ by SAS.

Because of the triangle congruence, $\angle CPM = \angle DAM$. This means that the transversal AP makes equal angles with PC and AD so PC will be parallel to AD. The line BC is another line through C that is parallel to AD so it is the same as line PC, so P lies on the line containing B, C, and N.

Again, by the congruence of the triangles, $CP = AD$ and $AD = BC = BN$, so if we use the magic ruler to divide PN into three equal parts, the division points must correspond to the missing points B and C. By extending CM and measuring off an additional length of CM on the other side of M, Laura can construct the final missing point D. ■



Note: other constructions are also possible.

5 Let $x_1, x_2, \ldots, x_k$ be a sequence of integers. A rearrangement of this sequence (the numbers in the sequence listed in some other order) is called a **scramble** if no number in the new sequence is equal to the number originally in its location. For example, if the original sequence is $1,3,3,5$ then $3,5,1,3$ is a scramble, but $3,3,1,5$ is not.

A rearrangement is called a **two-two** if exactly two of the numbers in the new sequence are each exactly two more than the numbers that originally occupied those locations. For example, $3,5,1,3$ is a two-two of the sequence $1,3,3,5$ (the first two values 3 and 5 of the new sequence are exactly two more than their original values 1 and 3).

Let $n \geq 2$. Prove that the number of scrambles of

$$1,1,2,3,\ldots,n-1,n$$

is equal to the number of two-twos of

$$1,2,3,\ldots,n,n+1 \quad .$$

(Notice that both sequences have $n+1$ numbers, but the first one contains two 1s.)

Solution: For the scrambles, we need to choose two locations from the $n-1$ numbers $2,3,\ldots,n$ to be occupied by the two 1s. Once this has been done, we are left with $n-1$ numbers, exactly two of which (the numbers whose locations were occupied by the 1s) can be placed freely while all the rest have exactly one location they cannot occupy.

For the two-twos, we need to choose two locations from the $n-1$ numbers $1,2,\ldots,n-1$ to be occupied by a number two greater than before; the list ends with $n-1$ since the n and $n+1$ spots don't have a number that is two greater than them. Then, we have $n-1$ remaining numbers, exactly two of which (1 and 2) can be placed freely while all the rest have exactly one location (the location two less than their value) they cannot occupy.

Notice that although the particular locations are different in the two descriptions above, the mechanics of making the selections are identical: Choose two from a particular subset of $n-1$ of the $n+1$ locations and fill them with particular items. Next fill the remaining slots with the remaining items such that two of the remaining items can go anywhere and each of the others is excluded from exactly one particular location.

Since the rearrangment process is identical in both cases, the number of scrambles and two-twos must be equal. ■

The calculation of the actual number of such scrambles or two-twos for a particular n is a bit difficult, but it is documented in the Online Encyclopedia of Integer Sequences: http://oeis.org/A105927.

6 Given a segment AB in the plane, choose on it a point M different from A and B. Two equilateral triangles $\triangle AMC$ and $\triangle BMD$ in the plane are constructed on the same side of segment AB. The circumcircles of the two triangles intersect in point M and another point N. (The **circumcircle** of a triangle is the circle that passes through all three of its vertices.)

(a) Prove that lines AD and BC pass through point N.

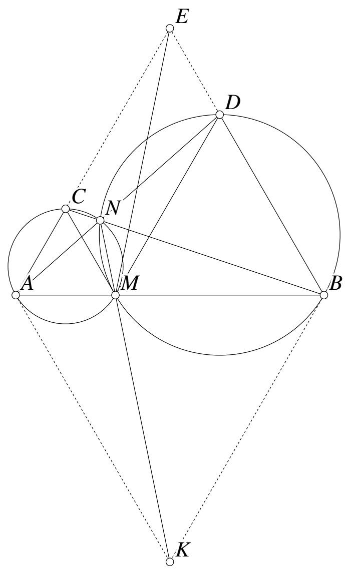
(b) Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

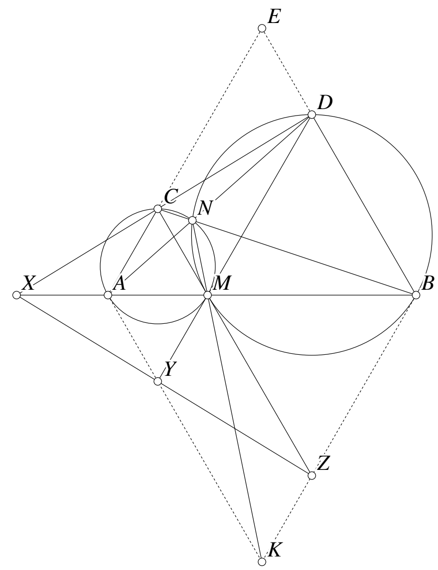
Solution: (a) It is not hard to show that point N is on the same side of segment AB as the two triangles, and that N is inside $\angle CMD$ so that $\{A,M,N,C\}$, and $\{B.M,N,D\}$, are arranged in these orders correspondingly on the circumcircles, as shown on the picture. (The reason is essentially that side MC is tangent to the circumcircle of $\triangle BMD$, because of the angle it makes with AB.) Since A, M, N and C are concyclic, and C and N are on the same side of line AB, $\angle ANM = \angle ACM = 60^\circ$. Since B, M, N and D are concyclic, and B and N are on opposite sides of chord MD, $\angle MND = 180^\circ - \angle MBD = 180^\circ - 60^\circ = 120^\circ$. Thus, the sum $\angle ANM + \angle MND = 60^\circ + 120^\circ = 180^\circ$, which proves that A, N, and D lie on a line. One can prove analogously that B, N, and C also lie on a line. ■

(b) Extend sides AC and BD until they intersect in point E, thereby creating another equilateral $\triangle ABE$. Reflect $\triangle ABE$ to $\triangle ABK$ across line AB. Note that point K is fixed, regardless of the chosen point M. We claim that line NM will always pass through point K.

Proof of Claim: To show that N, M and K lie on a line, it suffices to show that $\angle KMB = \angle AMN$. To this end, note that because of the reflection, $\angle KMB = \angle EMB$. From the External Angle Theorem applied to $\triangle AME$, we have $\angle EMB = \angle MAE + \angle AEM = \angle AMC + \angle EAD$; the latter is true because $\angle MAE = \angle AMC = 60^\circ$ and $\angle AEM = \angle EAD$ from the isosceles trapezoid $AMDE$. Finally, $\angle EAD = \angle CAN = \angle CMN$ from inscribed angles in the circumcircle of $\triangle AMC$. Putting everything together yields $\angle EMB = \angle AMC + \angle CMN = \angle AMN$. Thus, indeed, N, M and K are collinear. ■

Alternative Proof of Claim: We already know that AND and BNC are lines. We need to show that line MK also passes through N, i.e., that lines AD, BC and KM are concurrent, or in other words, that these lines are "perspective from point N". According to Desargues's Theorem, this is true if and only if the corresponding triangles $\triangle ABK$ and $\triangle DCM$ are "perspective from the line" formed by the intersection of their corresponding sides.[1] Let lines AB and DC intersect in point X, lines AK and DM intersect in point Y, and lines BK and CM intersect in point Z. Thus, it suffices to show that XYZ is also a line. However, note that Y and Z are the reflections of C and D across AB (because $\triangle AMY$ and $\triangle BMZ$ are again equilateral). Hence, line XCD reflects to *line* XYZ, proving our statement. ■





Note: a number of other solutions to the problem were provided by BAMO 2012 participants, including solutions using inversion in the plane, radical axes, and other extra constructions.

Note: This problem was inspired by a problem on the first International Mathematical Olympiad in 1959, where equilateral triangles are replaced by squares. In fact, a more general version that incorporates both problems is the following:

Generalization: Given a segment AB and a pont M inside of it, construct circle ω_l centered at O_l passing through A and M and ω_r centered at O_r passing through M and B so that O_l and O_r are on the same side of AB and $\angle AO_lM = \angle MO_rB = 2x$. Then ω_l andω_r intersect at M and another point N. Extend AN until it intersects ω_r again at a point D. Prove that $\angle DBA = x$, and moreover, all lines NM pass through the same point K in the plane. (Note that for triangles we have $x = 60^\circ$, and for squares we have $x = 45^\circ$.)

Solution to Generalization: As above, N is on the same side of AB as O_l and O_r.

For the first part, $\angle ANM = x$ because it spans the arc AM; hence $\angle MND = 180^\circ - x$. As $MNDB$ is cyclic, we have $\angle MBD = x$.

For the second part, $\angle ANB = \angle ANM + \angle MNB = x + x = 2x$, so that N is on the circle ω passing through A and B for which the arc AB spans an angle of $2x$. Consider the point K of ω which is on the other side of AB from N and is such that $KA = KB$. Then $\angle KNA = \angle KNB$ as they span equal arcs, implying that KN passes through M. ■

[1]Two triangles are *perspective from a point* if the corresponding vertices of the two triangles form three lines intersecting in a single point. Two triangles are *perspective from a line* if the corresponding sides of the two triangles (or their continuations) intersect in three points that lie on a line. Desargues' Theorem states that two triangles are perspective from a point if and only if they are perspective from a line. See, for example, "Geometric Puzzles and Constructions – Six Classical Geometry Theorems" in *Mathematical Adventures For Students and Amateurs*, edited by David F. Hayes and Tatiana Shubin, published by the Mathematical Association of America.

7 Find all nonzero polynomials $P(x)$ with integer coefficients that satisfy the following property: whenever a and b are relatively prime integers, then $P(a)$ and $P(b)$ are relatively prime as well. Prove that your answer is correct. (Two integers are **relatively prime** if they have no common prime factors. For example, -70 and 99 are relatively prime, while -70 and 15 are not relatively prime.)

Solution: Answer: $P(x) = \pm x^n$ for each integer $n \geq 0$.

It is evident that these polynomials meet the condition, since the only possible prime factors of $P(a)$ are the prime factors of a, so if a, b have no prime factors in common, $P(a), P(b)$ can't either.

Consider any polynomial P not of this form; we show that it does not meet the condition. Write

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0.$$

Replacing $P(x)$ by $-P(x)$ if necessary, we may assume $c_n > 0$.

Suppose that $c_n = 1$ and the next nonzero coefficient c_k is negative. Then we have $x^{n-1} < P(x) < x^n$ for all large enough x. In all other cases, we have $x^n < P(x) < x^{n+1}$ for all large enough x. In either situation, if we choose q to be a large enough prime, then $P(q)$ is a positive integer lying betwen two consecutive powers of q. In particular, $P(q)$ cannot itself be a power of q, so it must have some other prime factor $r \neq q$.

Then the numbers q and $q + r$ are relatively prime. But since

$$r = (q + r) - q \mid P(q + r) - P(q),$$

both $P(q)$ and $P(q + r)$ are divisible by r, so they are not relatively prime. Hence, the polynomial P does not satisfy the required condition.

■