23rd Bay Area Mathematical Olympiad

Problems and Solutions

March 2, 2022

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

A I have cards with whole numbers written on them. I can put them in various orders in a row, next to each other, to create a single long number. For example, if my cards were 3, 6, and 18, some of the numbers I could make are 3618, 6183, and 1863. However, 3861 and 8136 would **not** be allowable arrangements, as the number "18" is written on one card: it cannot be split up and its digits cannot be switched. Note that cards can only be arranged in a row; they may not be turned over or manipulated in other ways, and thus, 6^{318} and 3918 are **not** allowable arrangements of 3, 6, and 18.

If I have 100 cards with all the numbers 1 through 100 on them, how should I put them in order to create the largest possible number? Describe exactly how your answer is constructed. No formal proof is required.

Solution. Our final answer will have the same number of digits no matter what order we put the cards in. When comparing two integers with the same number of digits, we can determine which is larger by finding the first position (from left to right) where their digits differ; the larger number is the one whose digit in that position is larger. For example, 9751 is bigger than 9748 because their first differing digits are the 5 and 4 in the tens place, and $5 > 4$. (Note that this is the same principle as alphabetic order!)

Thus, our best strategy is to choose the leftmost card first, then the next card to the right, and so on, in such a way that we always make the next digit as large as possible; if there is more than one card which will accomplish this, we should break the tie so as to make the digit after that as large as possible, and so on. For this reason, all the cards whose first digit is 9 should precede all the other cards. 99 should go before 98, which should go before 97, and so on. It takes a little more thought to figure out where the 9 card should go. It can go before or after 99, but should go before 98, to maximize the number of consecutive 9s before we must have a digit smaller than 9.

Applying similar reasoning to all the cards, we get the ordering

$$\underline{9\ 99}\ 98\cdots 90\ 89\ \underline{8\ 88}\ 87\cdots 80\ 79\ 78\ \underline{7\ 77}\ 76\cdots 12\ \underline{1\ 11}\ 10\ 100,$$

where the underlined pairs can appear in either order without affecting the final number.[1] ■

[1] Because we always make it our highest priority to make the very next digit as large as possible, this is an example of what computer programmers call a *greedy algorithm*.

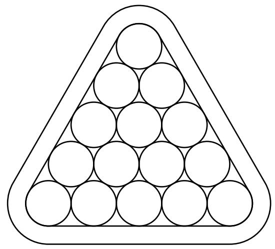
B You are bargaining with a salesperson for the price of an item. Your first offer is \$$a$ and theirs is \$$b$. After you raise your offer by a certain percentage and they lower their offer by the same percentage, you arrive at an agreed price. What is that price, in terms of a and b?

Solution. Let r be the percentage mentioned in the problem, expressed as a decimal (e.g. if the percentage is 20% then $r = 0.2$). If p is the final price, then $p = a(1+r)$ and $p = b(1-r)$. We set these equal to each other and solve:

$$\begin{aligned} a(1+r) &= b(1-r) \\ a+ar &= b-br \\ (a+b)r &= b-a \\ r &= \frac{b-a}{a+b} \end{aligned}$$

Then $p = a\left(1+\frac{b-a}{a+b}\right) = a\cdot\frac{a+b+b-a}{a+b} = \frac{2ab}{a+b}$, so this is the agreed price. (This expression is also known as the harmonic mean of a and b.) ■

C/1 The game of pool includes 15 balls that fit within a triangular rack as shown:



Seven of the balls are "striped" (not colored with a single color) and eight are "solid" (colored with a single color). Prove that no matter how the 15 balls are arranged in the rack, there must always be a pair of striped balls adjacent to each other.

Solution. We may dissect the rack into six regions in the following manner (indicated by colors, as well as by numbers in case you are viewing this solution in black and white):

Since there are seven striped balls, at least one of the six regions will contain a pair of striped balls.[2] But every ball is adjacent to the other balls in its region. Therefore, some two striped balls must be adjacent to each other. ∎

Alternative solution. Suppose (aiming for a contradiction) that no two striped balls are adjacent to each other. Call the 12 balls along the outer edges of the triangle *outer balls*, and the 3 in the center *inner balls*.

At most 1 inner ball can be striped, since the inner balls are all mutually adjacent. Thus, *at least* 6 outer balls must be striped.

On the other hand, the 12 outer balls form a ring, so *at most* 6 of them can be striped, which means *exactly* 6 must be striped; and this can be achieved only if striped and non-striped balls alternate around the outer ring. In either of the two such arrangements, the striped inner ball is adjacent to a striped outer ball, and we have a contradiction. ∎

[2]Here we are using the *pigeonhole principle*, a highly useful idea in combinatorics. The pigeonhole principle says that if $n+1$ pigeons take up residence in n holes, then it is inevitable that one of the holes will contain at least two pigeons. Many problems can be solved by cleverly identifying "pigeons" and "holes" to which to apply the principle.

D/2 Suppose that p, $p+d$, $p+2d$, $p+3d$, $p+4d$, and $p+5d$ are six prime numbers, where p and d are positive integers. Show that d must be divisible by 2, 3, and 5.

Solution. In general, let p be a prime number and let d be an integer not divisible by p. We claim that among p consecutive terms in an arithmetic progression with common difference d, no two terms leave the same remainder when divided by p. To illustrate what this is saying, consider the case of $p = 5$, $d = 12$, and the 5-term progression $4, 16, 28, 40, 52$. When we divide these terms by 5, the remainders are $4, 1, 3, 0, 2$. As claimed, these remainders are all different.

The claim implies something further: *all* possible remainders from 0 to $p-1$ appear, and since 0 appears, one of the original terms is a multiple of p.

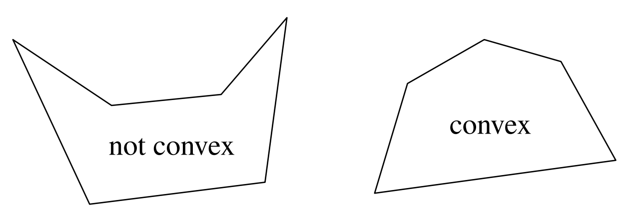
Here is a proof of the claim, arguing by contradiction. Suppose two of the terms, k terms apart, left equal remainders when divided by p. The difference of those terms, kd, must then be a multiple of p, which means that at least one of k, d must be divisible by p. Because we are only considering p consecutive terms, $k < p$. Thus k is not divisible by p. But we also assumed that d is not divisible by p, so we have a contradiction. This proves the claim.

Turning now to the problem, suppose we have six primes in increasing arithmetic progression. If 2 does not divide d, then three of the terms must be even (namely, one of the first two, one of the next two, and one of the last two). Of these three even terms, only one can be 2 itself, so at least two of them must be composite.

In a similar manner, if 3 does not divide d, then two of the terms must be divisible by 3 and at least one of these must be composite. Therefore d must be divisible by 2 and 3.

Finally, if d is not divisible by 5, then one of the last five terms is a multiple of 5. These terms are all greater than or equal to $1+6$, so the multiple of 5 cannot be 5 itself, and must be composite. We conclude that d is divisible by 5 as well, and we are finished. ■

E/3 A polygon is called *convex* if all of its internal angles are smaller than $180°$. Here are examples of nonconvex and convex polygons:



Given a convex polygon, prove that one can find three distinct vertices A, P, and Q, where PQ is a side of the polygon, such that the perpendicular from A to the line PQ meets the segment PQ (possibly at P or Q).

Solution. Let PQ be the longest side. (Or one of the longest sides, if there are ties.) Now consider the strip in the plane traced out by translating segment PQ perpendicular to its original position. (This is the set of points T such that the perpendicular from T to line PQ actually intersects segment PQ.) Since a polygon is a closed loop, this strip must cross the polygon a second time, other than along segment PQ. The strip must contain another vertex A of the polygon, since if it does not, then it must pass through the interior of another side, which would then be longer than side PQ. This is impossible, since no side of the polygon is longer than side PQ. Thus we have found a vertex A with the desired property. ■

4 Ten birds land on a 10-meter-long wire, each at a random point chosen uniformly along the wire. (That is, if we pick out any x-meter portion of the wire, there is an $\frac{x}{10}$ probability that a given bird will land there.)

What is the probability that every bird sits more than one meter away from its closest neighbor?

Solution 1. Labelling the birds by the integers $0,1,\ldots,9$, we can refer to their positions along the wire by real numbers $x_0,x_1,\ldots,x_9$ where $0 \le x_i < 10$ (there is zero probability that any bird lands *exactly* at 10, so it doesn't matter if we exclude it from the interval).

One configuration that works has the birds at $0.1, 1.2, 2.3, 3.4, 4.5, 5.6, 6.7, 7.8, 8.9, 9.95$. The idea is that the integer parts are 0, 1, 2, 3, 4, 5, 6, 7, 8,9, and the fractional parts are *strictly increasing*. Let's formalize this idea.

Given a real x, there is a unique way to write $x = [x] + \{x\}$, where $[x]$ is an integer (the integer part of x) and $0 \le \{x\} < 1$ (the fractional part of x). To choose a random number x uniformly from $[0,10)$, it is equivalent to do the following: choose an integer n from $\{0,1,\ldots,9\}$ and choose a real number t from $[0,1)$ (both uniformly at random), then define $x = n+t$. This clearly implies that $[x] = n$ and $\{x\} = t$, so in this sense, the integer and fractional parts of x are independent. For any distinct birds i and j, the probability that $\{x_i\} = \{x_j\}$ is zero, so we may assume that the fractional parts of their positions $x_0,x_1,\ldots,x_9$ are all distinct.

If two distinct birds i and j satisfy $[x_i] = [x_j]$, then it is clear that

$$\left|x_i - x_j\right| = \left|\{x_i\} - \{x_j\}\right| < 1,$$

so they will sit less than one meter apart. So if the birds are to sit far enough apart, they must have distinct integer parts. There are 10 birds and 10 potential integer parts, so an appropriate labelling of the birds will then yield $[x_i] = i$ for each $i = 0,1,\ldots,9$. This clearly implies that $x_0 < x_1 < \cdots < x_9$, so we just need to check the distance between successive birds:

$$x_i - x_{i-1} = \big(i + \{x_i\}\big) - \big(i - 1 + \{x_{i-1}\}\big) = 1 + \{x_i\} - \{x_{i-1}\}.$$

This distance is greater than 1 if and only if $\{x_i\} > \{x_{i-1}\}$. Thus each bird sits more than one meter away from its closest neighbor if and only if the birds can be labelled by the integers $0,1,\ldots,9$ in such a way that

(a) the integers parts *are* the labels: $[x_i] = i$ for each $i = 0,1,\ldots,9$;

(b) and the fractional parts are increasing: $\{x_0\} < \{x_1\} < \cdots < \{x_9\}$.

But since we assumed that the fractional parts are all distinct, there is a unique way to label the birds so that the fractional parts are increasing. Hence, they are far enough apart if and only if $[x_i] = i$ for all $i = 0,1,\ldots,9$ (in this labelling). Since the integer and fractional parts are of each x_i are independent, the probability that $[x_i] = i$ is $1/10$, for any individual $i = 0,1,\ldots,9$. Therefore, the desired probability is precisely $1/10^{10}$. ■

Solution 2. (Sketch) Another argument is to show that there is a one-to-one correspondence between any configuration of 10 birds on a 10-meter wire where each bird is more than one meter from its closest neighbor, and any configuration of 10 birds on a 10-meter wire where all ten birds lie between position 0 and position 1.

5 Sofiya and Marquis are playing a game. Sofiya announces to Marquis that she's thinking of a polynomial of the form $f(x) = x^3 + px + q$ with three integer roots that are not necessarily distinct. She also explains that all of the integer roots have absolute value less than (and not equal to) N, where N is some fixed number which she tells Marquis. As a "move" in this game, Marquis can ask Sofiya about any number x and Sofiya will tell him whether $f(x)$ is positive, negative, or zero. Marquis's goal is to figure out Sofiya's polynomial.

If $N = 3 \cdot 2^k$ for some positive integer k, prove that there is a strategy which allows Marquis to identify the polynomial after making at most $2k+1$ "moves".

Solution. We will detail Marquis's strategy. Let's begin with a few algebraic observations.

Let the roots of f be r_1, r_2, r_3 (including repeated roots as appropriate). Then

$$f(x) = c(x-r_1)(x-r_2)(x-r_3).$$

Since $f(x) = x^3 + px + q$, we can see by equating coefficients of x^3 that $c = 1$. Thus, if we can determine r_1, r_2, r_3, that is enough to uniquely identify f. We will therefore focus on finding the roots.

In $x^3 + px + q$, the coefficient of x^2 is 0, while in the expansion of $(x-r_1)(x-r_2)(x-r_3)$, the coefficient of x^2 is $-(r_1+r_2+r_3)$. It follows that the sum of the roots is 0.[3]

We also observe that we know the signs of $f(N)$ and $f(-N)$ without asking Sofiya. Since $|r_1|, |r_2|, |r_3| < N$, we have

$$f(N) = (N-r_1)(N-r_2)(N-r_3) > 0,$$
$$f(-N) = (-N-r_1)(-N-r_2)(-N-r_3) < 0.$$

We use our first question to ask Sofiya for the sign of $f(0)$, then proceed to one of three cases based on her answer.

First case: $f(0) = 0$. In this case, the roots are of the form $-r, 0, r$ where $0 \le r < N$. We know that $f(x)$ must change sign at $x = r$, but nowhere else to the right of $x = 0$. Thus if $f(x)$ has the same sign at a and b where $0 < a < b$, the interval $[a, b]$ cannot contain r.

We adopt a binary search strategy to find r. Initially we know that r is in the interval $[0, N-1]$. Our goal will be to reduce the candidates by at least half with each query. The first step is to ask Sofiya for the sign of $f(\frac{1}{2}N)$. We previously noted that $f(N) > 0$. Thus, if $f(\frac{1}{2}N) > 0$, r is in the interval $[0, \frac{1}{2}N - 1]$; if $f(\frac{1}{2}N) < 0$, r is in the interval $[\frac{1}{2}N + 1, N-1]$; and if $f(\frac{1}{2}N) = 0$, then $r = \frac{1}{2}N$.

At each subsequent step, if our previous queries have confined r to the interval $[a, b]$, we next ask for the sign of f at $\frac{a+b}{2}$, and Sofiya's answer tells us whether r is less than, equal to, or greater than $\frac{a+b}{2}$.

In this way, each query reduces the candidates by at least half until we have found r. Since we started with $N = 3 \cdot 2^k < 2^{k+2}$ possibilities for r, this strategy catches r after at most $k+1$ queries, not counting the original query which determined that $f(0) = 0$. Hence we have identified f with a total of $k+2$ queries, which is as good as or better than the $2k+1$ required. This concludes the first case.

[3] This is an instance of a more general idea which sometimes goes under the name of *Vieta's formulas*. In general, if a polynomial of degree n has leading term 1 and roots $r_1, \ldots, r_n$, the coefficient of x^k in its expansion is $(-1)^{n-k} e_{n-k}(r_1, \ldots, r_n)$, where e_{n-k} is the *elementary symmetric polynomial* given by

$$e_{n-k}(x_1, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_{n-k} \le n} x_{i_1} x_{i_2} \cdots x_{i_{n-k}}.$$

The case we have used in the solution is $n = 3$, $k = 2$.

Second case: $f(0) < 0$. In this case, $f(x)$ has opposite sign at $x = 0$ and $x = N$, so f must have an odd number of roots in the interval $[1, N-1]$. But it cannot be true that all three roots lie in that interval, since the sum of the roots is 0. Thus there is exactly one root in the interval $[1, N-1]$, and the other two roots are negative.

We use our second question to ask the sign of $f(\frac{2}{3}N)$, that is, $f(2^{k+1})$. We proceed to one of three subcases based on Sofiya's answer.

First subcase: $f(\frac{2}{3}N) = 0$. Then we have found the positive root (it's $\frac{2}{3}N$). The other two roots are negative and add up to $-\frac{2}{3}N$, so the greater of them lies in the interval $[-\frac{1}{3}N, -1]$, which contains 2^k candidates. A binary search finds it in at most k additional queries, and then we know all the roots. We have made at most $k+2$ queries—as good as or better than required.

Second subcase: $f(\frac{2}{3}N) < 0$. In this case, since $f(0)$ and $f(\frac{2}{3}N)$ are of the same sign, we know there is no root in $[0, \frac{2}{3}N]$; the positive root is in the interval $[\frac{2}{3}N + 1, N-1]$, which contains $2^k - 1 < 2^k$ candidates. A binary search determines the positive root with at most $k-1$ more queries. Let this root be r. The other two roots are negative and add up to $-r$, so the greater of them lies in the interval $[-\frac{1}{2}r, -1]$, which contains at most $\frac{1}{2}(N-1) < 3 \cdot 2^{k-1} < 2^{k+1}$ candidates. A binary search finds it in at most k additional queries, and then we know all the roots. We have made at most $2 + (k-1) + k = 2k+1$ queries, as required.

Third subcase: $f(\frac{2}{3}N) > 0$. In this case, since $f(0)$ and $f(\frac{2}{3}N)$ are of opposite sign, we know the positive root is in the interval $[1, \frac{2}{3}N - 1]$, which contains $2^{k+1} - 1 < 2^{k+1}$ candidates. A binary search determines the positive root with at most k more queries. Let this root be r. The other two roots are negative and add up to $-r$, so the greater of them lies in the interval $[-\frac{1}{2}r, -1]$, which contains $\lfloor \frac{1}{2}r \rfloor < \lfloor \frac{1}{2}(\frac{2}{3}N - 1) \rfloor < 2^k$ candidates. A binary search finds this root in at most $k-1$ additional queries, and then we know all the roots. We have made at most $2 + k + (k-1) = 2k+1$ queries, as required.

This concludes the second case.

Third case: $f(0) > 0$. This case is essentially the same as the second case, except that instead of one positive root and two negative roots, there are two positive roots and one negative root. We may proceed in the same way as in the second case, reversing the signs of all x in queries to Sofiya.

Thus, we have dealt with all cases, concluding the solution. ■