25th Bay Area Mathematical Olympiad

Problems and Solutions

Mar 6, 2024

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

Problems

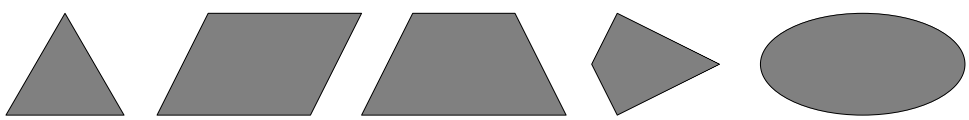
A A school needs to elect its president. The school has 121 students, each of whom belongs to one of two tribes: Geometers or Algebraists. Two candidates are running for president: one Geometer and one Algebraist. The Geometers vote only for Geometers and the Algebraists only for Algebraists. There are more Algebraists than Geometers, but the Geometers are resourceful. They convince the school that the following two-step procedure is fairer:

(a) The school is divided into 11 groups, with 11 students in each group. Each group elects a representative for step 2.

(b) The 11 elected representatives elect a president.

Not only do the Geometers manage to have this two-step procedure approved, they also volunteer to assign the students to groups for step 1. What is the minimum number of Geometers in the school that guarantees they can elect a Geometer as president? (In any stage of voting, the majority wins.)

B Amelia's mother proposes a game. "Pick two of the shapes below," she says to Amelia. (The shapes are an equilateral triangle, a parallelogram, an isosceles trapezoid, a kite, and an ellipse. These shapes are drawn to scale.) Amelia's mother continues: "I will draw those two shapes on a sheet of paper, in whatever position and orientation I choose, without overlapping them. Then you draw a straight line that cuts both shapes, so that each shape is divided into two congruent halves."



Which two of the shapes should Amelia choose to guarantee that she can succeed? Given that choice of shapes, explain how Amelia can draw her line, what property of those shapes makes it possible for her to do so, and why this would not work with any other pair of these shapes.

C/1 Sugar Station sells 44 different kinds of candies, packaged one to a box. Each box is priced at a positive integer number of cents, and it costs \$1.51 to buy one of every kind. (There is no discount based on the number of candies in a purchase.) Unfortunately, Anna only has \$0.75.

(a) Show that Anna can buy at least 22 boxes, each containing a different candy.

(b) Show that Anna can do even better, buying at least 25 boxes, each containing a different candy.

D/2 Sasha wants to bake 6 cookies in his 8 inch $\times$ 8 inch square baking sheet. With a cookie cutter, he cuts out from the dough six circular shapes, each exactly 3 inches in diameter. Can he place these six dough shapes on the baking sheet without the shapes touching each other? If yes, show us how. If no, explain why not. (Assume that the dough does not expand during baking.)

E/3 Let S_n be the sum of the first n prime numbers. For example,

$$S_5 = 2+3+5+7+11 = 28.$$

Does there exist an integer k such that $S_{2023} < k^2 < S_{2024}$?

4 Find all polynomials f that satisfy the equation

$$\frac{f(3x)}{f(x)} = \frac{729(x-3)}{x-243}$$

for infinitely many real values of x.

5 An underground burrow consists of an infinite sequence of rooms labeled by the integers $(\ldots,-3,-2,-1,0,1,2,3,\ldots)$. Initially, some of the rooms are occupied by one or more rabbits.

Each rabbit wants to be alone. Thus, if there are two or more rabbits in the same room (say, room m), half the rabbits (rounding down) will flee to room $m-1$, and half (also rounding down) to room $m+1$. Once per minute, this happens simultaneously in all rooms that have two or more rabbits. For example, if initially all rooms are empty except for 5 rabbits in room #12 and 2 rabbits in room #13, then after one minute, rooms #11– #14 will contain 2, 2, 2, and 1 rabbits, respectively, and all other rooms will be empty.

Now suppose that initially there are $k+1$ rabbits in room k for each $k = 0,1,2,\ldots,9,10$, and all other rooms are empty.

(a) Show that eventually the rabbits will stop moving.

(b) Determine which rooms will be occupied when this occurs.

Solutions

A **Solution:** To ultimately win the election in step 2, the Geometers must have 6 seats among the 11 representatives who elect the president.

Moving back to step 1, the Geometers need 6 out of 11 votes in a group to elect a representative from that group. Thus, it takes $6 \times 6 = 36$ Geometers to elect 6 representatives and thereby win the presidency.

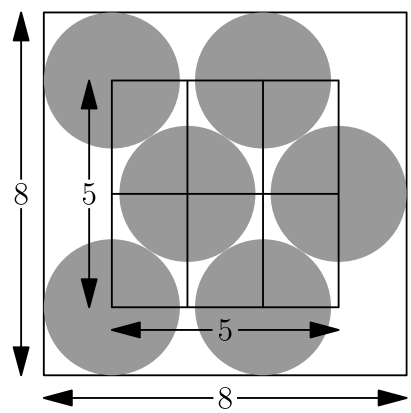
B **Solution:** Amelia should choose the parallelogram and the ellipse, which are then simultaneously bisected into congruent halves by the line through their centers. This works because these two shapes have half-turn rotational symmetry.

Each of the other three shapes has only a finite number of lines cutting it into two congruent parts. The equilateral triangle has three such lines—the three axes of symmetry. The trapezoid and the kite have just one axis of symmetry each. If Amelia chooses one of these three shapes, her mother can always position it in such a way that its axis or axes of symmetry do not coincide with any axis of symmetry of the other figure. Indeed, if we call the intersection of the diagonals of the trapezoid (respectively, the kite) the center of that figure, then one easy way for Amelia's mom to construct an impossible configuration is to make sure that the axes of symmetry of one shape do not pass through the center of the other shape.

C/1 Solution: For part (a), pick boxes containing 22 different candies chosen at random. Let their total cost be m cents. If $m \le 75$, Anna can buy these candies. Otherwise, $m \ge 76$. In this case, the *other* 22 candies have a total cost of $151 - m \le 75$ cents, so Anna can buy those candies instead.

For part (b), rank the candies from most to least expensive and consider the 19$^{\text{th}}$ candy in the ranking. If this candy costs 4 cents or more, then the first 19 candies cost at least 76 cents and so the last 25 candies cost at most 75 cents. But if the 19$^{\text{th}}$ most expensive candy costs 3 cents or less, then so do the remaining 25 candies in the list, so the last 25 candies again cost at most 75 cents.

D/2 Solution: Yes, but just barely! The centers of the cookies must lie within the middle 5×5 square. Divide this square in half one way and in thirds the other way, creating a grid of six $\frac{5}{2} \times \frac{5}{3}$ rectangles, and place cookie centers at alternate intersections of this grid as shown.



Then the distance between nearest neighboring cookie centers is

$$\sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{\sqrt{325}}{6} > 3,$$

so the cookies do not touch or overlap.

E/3 Solution: Claim: There exists an integer k such that $S_{2023} < k^2 < S_{2024}$.

Proof: Let k be the smallest integer such that $S_{2023} < k^2$. Note that

$$k^2 = 1 + 3 + 5 + \cdots + O_k,$$

where $O_k = 2k - 1$, the k^{th} odd number. Furthermore, observe that $O_k \le p_{2023}$, the 2023$^{\text{rd}}$ prime. This follows from the fact that $S_{2023} = 2 + 3 + 5 + 7 + 11 + \cdots + p_{2023}$ is a sum of 2023 primes, and if we compare this with the sum of all the odd integers up to p_{2023}, this second sum is surely larger, since only the first term is smaller (1 versus 2), and otherwise it contains every number in the first sum, plus many more odd numbers (since there are gaps in the primes). Note that this is a very crude estimate!

By the minimality of k, we know that

$$(k-1)^2 = 1 + 3 + 5 + \cdots + (O_k - 2) \le S_{2023}.$$

Adding O_k to both sides yields

$$\begin{aligned}1+3+5+\cdots+(O_k-2)+O_k &= k^2\\ &\le S_{2023}+O_k\\ &\le S_{2023}+p_{2023}\\ &< S_{2023}+p_{2024}\\ &= S_{2024},\end{aligned}$$

and we are done.

Alternative version: We argue by contradiction. Suppose there is no perfect square between S_{2023} and S_{2024}. Then there is some positive integer r such that

$$r^2 \le S_{2023} < S_{2024} \le (r+1)^2.$$

Consequently, $S_{2024}-S_{2023}$ (which is the 2024$^{\text{th}}$ prime) is at most $(r+1)^2-r^2=2r+1$, and S_{2023} is at most the sum of all primes no larger than $2r$. This sum is made up of 2 and some, but not all, of the odd numbers up to $2r$ (in particular, 1 and 9 are missing). Thus,

$$\begin{aligned}S_{2023} &\le (1+3+5+7+9+\cdots+(2r-1))-(1+9)+2\\ &= r^2-8,\end{aligned}$$

which contradicts the premise that $r^2 \le S_{2023}$. Therefore, there must be a perfect square between S_{2023} and S_{2024}.

4 Solution: The above equation holds for infinitely many x if and only if

$$(x-243)f(3x) = 729(x-3)f(x)$$

for all $x \in \mathbb{C}$, because $(x-243)f(3x)-729(x-3)f(x)$ is a polynomial, which has infinitely many zeroes if and only if it is identically 0.

We now plug in different values to find various zeroes of f:

$$\begin{aligned}x=3 &\implies f(9)=0\\ x=9 &\implies f(27)=0\\ x=27 &\implies f(81)=0\\ x=81 &\implies f(243)=0\end{aligned}$$

We may write $f(x)=(x-243)(x-81)(x-27)(x-9)p(x)$ for some polynomial $p(x)$. We want to solve

$$\begin{aligned}&(x-243)(3x-243)(3x-81)(3x-27)(3x-9)p(3x)\\ &= 729(x-3)(x-243)(x-81)(x-27)(x-9)p(x).\end{aligned}$$

Dividing out common factors from both sides, we get $p(3x)=9p(x)$, so the polynomial p is homogeneous of degree 2. Therefore $p(x)=ax^2$ and thus $f(x)=a(x-243)(x-81)(x-27)(x-9)x^2$, where $a\in\mathbb{C}$ is arbitrary. $\square$

Alternative solution. As above, we wish to find polynomials f such that

$$(x-243)f(3x) = 729(x-3)f(x).$$

Suppose f is such a polynomial and let Z be its multiset of zeroes.

Both $(x-243)f(3x)$ and $729(x-3)f(x)$ have the same multiset of zeroes, i.e. $\{243\} \cup \frac{1}{3}Z = \{3\} \cup Z$ or $\{729\} \cup Z = \{9\} \cup 3Z$. Thus

$$\begin{aligned} 9 \in Z &\Rightarrow 27 \in 3Z \\ &\Rightarrow 27 \in Z \\ &\Rightarrow 81 \in 3Z \\ &\Rightarrow 81 \in Z \\ &\Rightarrow 243 \in 3Z \\ &\Rightarrow 243 \in Z. \end{aligned}$$

Let Y be the unique multiset with $Z = \{9, 27, 81, 243\} \cup Y$. This gives

$$\{9, 27, 81, 243, 729\} \cup Y = \{9, 27, 81, 243, 729\} \cup 3Y,$$

so $Y \subset \mathbb{C}$ is a finite multiset, invariant under multiplication by 3. It follows that $Y = \{0, 0, \ldots, 0\}$ with some multiplicity k. Hence

$$f(x) = ax^k(x-9)(x-27)(x-81)(x-243)$$

for some constant $a \in \mathbb{C}$. Plugging this back into the original equation, we see that $k = 2$ and that $a \in \mathbb{C}$ can be arbitrary. □

5 Solution: First, we show that the process eventually stops.

Call a room *interior* if it is occupied, or if at least one room somewhere to its left and at least one room somewhere to its right are occupied.

We claim that it is not possible for a gap of two or more consecutive unoccupied interior rooms to ever appear, given that no such gap exists in the initial configuration. For consider the first moment when such a gap appears, and let A and B be the two leftmost rooms in that gap. Either A or B must have been occupied at the previous step. But when rabbits leave A, some of them must go to B, and vice versa, creating a contradiction. This proves the claim.

Rabbits move in pairs, from room k to rooms $k-1$ and $k+1$. Since $k+k = (k-1)+(k+1)$, the sum of the rabbits' room numbers is constant. However, the sum of their room numbers' *squares* increases at every step, since $(k-1)^2 + (k+1)^2 = k^2 + k^2 + 2$. This shows that the rabbits' configuration can never recur, neither exactly nor up to a shift.

Up to a shift, there are finitely many possible configurations of rabbits with no internal gaps of two or more rooms. Thus we have shown that the process terminates after a finite number of steps.

Now we will determine the final configuration.

We claim that the final configuration cannot have two empty interior rooms (consecutive or otherwise!). In particular, we will prove by strong induction on n that at no time do n consecutive interior rooms contain $n-2$ or fewer rabbits.

Base case ($n = 0$): In this case, the claim is that 2 consecutive interior rooms cannot contain 0 rabbits. We already proved this above.

Inductive step: Suppose the proposition has been proven up to some fixed n (inclusive). We wish to show that $n+1$ consecutive interior rooms cannot ever contain $n-1$ or fewer rabbits. Aiming for a contradiction, suppose that, at some time, some $n+1$ consecutive interior rooms do contain $n-1$ or fewer rabbits. We claim those rooms must contain $0, \underbrace{1, 1, \ldots, 1}_{n-1}, 0$ rabbits respectively.

Proof: Any other possible distribution among those rooms would include at least three empty rooms $0, \underbrace{\ldots\ldots}_{r \text{ rooms}}, 0, \underbrace{\ldots\ldots}_{\substack{n-r-2 \\ \text{rooms}}}, 0$. By the inductive hypothesis, there must be at least $r+1$ rabbits in the first $r+2$

rooms and at least $n-1-r$ rabbits in the last $n-r$ rooms. But that makes n rabbits, which is a contradiction. The claim is proved; let us proceed.

Let the two rooms on the ends of the $0,\underbrace{1,1,\ldots,1}_{n-1},0$ sequence be room k and room $k+n$. These rooms must have been occupied at some point (since they are interior). Without loss of generality, suppose room k contained a rabbit at least as recently as room $k+n$, and consider the last moment when room k was occupied. At the next moment, a rabbit must have fled from room k to room $k+1$. After that time, the total number of rabbits in rooms $k,\ldots,k+n$ cannot have changed. Thus, the state $0,\underbrace{1,1,\ldots,1}_{n-1},0$ was attained at that moment.

If at that moment rooms k and $k+n$ discharged simultaneously, then before they did so, rooms $k+1,k+2,\ldots,k+n-1$ must have contained at most $n-3$ rabbits. If room $k+n$ was already empty when room k discharged for the last time, then rooms $k+1,k+2,\ldots,k+n$ must have contained at most $n-2$ rabbits. Either way, the inductive hypothesis is violated, which completes our proof that the final configuration does not have two empty interior rooms.

We have determined that the final configuration consists of a sequence of consecutive rooms of which all, or all but one, are occupied by 1 rabbit each. There are 66 rabbits total. We also know that the sum of the rabbits' room numbers never changes from its initial value of $(0)(1)+(1)(2)+(2)(3)+\cdots+(10)(11)=440$.

To find a configuration meeting the requirements, we first consider whether the rabbits could be in 66 consecutive rooms. If so, their median room number would have to be $\frac{440}{66}=6\frac{2}{3}$, but this is neither an integer nor half an integer, so we can rule it out.

Thus the rabbits occupy some 67 consecutive rooms (except one), centered somewhere near room 7. The 67 rooms centered at room 7 are rooms $-26,\ldots,40$, with a sum of 469. Thus, to achieve a sum of 440, we can fill all these rooms except room 29. That's a possible final configuration.

Finally, we verify that this answer is unique. If the 67 consecutive rooms are farther to the left, then their sum is at most $(-27)+(-26)+\cdots+39=402$, and omitting one room from this sum can't get us to 440. If they are further to the right, then their sum is at least $(-25)+(-24)+\cdots+41=536$, which is likewise too high. Thus, the configuration we computed must be the correct one.