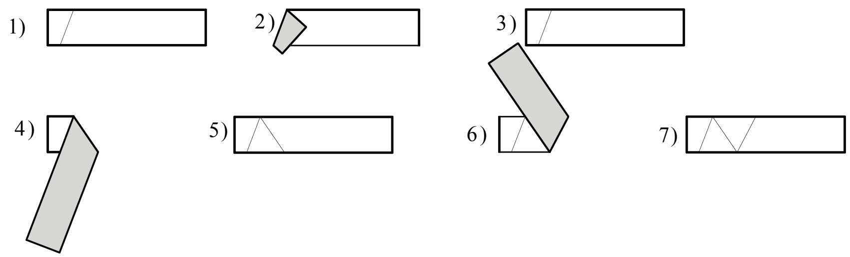
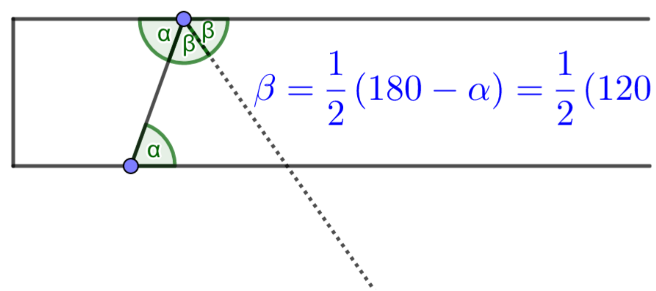
# Equilateral triangle algorithm?



## Not so fast!

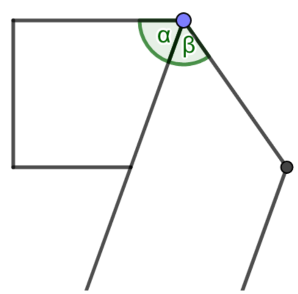
Let $x$ be the "error". Thus $\alpha = 60 + x$, and




$$\beta = \frac{1}{2}(180 - \alpha) = \frac{1}{2}(120 - x) = 60 - \frac{x}{2}$$

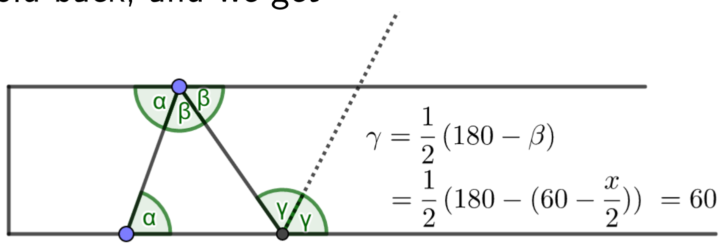
# Not so fast!

After folding, we get



# Not so fast!

Fold back, and we get




$$\gamma = \frac{1}{2}(180 - \beta)$$
$$= \frac{1}{2}(180 - (60 - \frac{x}{2})) \; = 60 + \frac{x}{4}$$

## Not so fast!

Thus, our angles are

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Thus, our angles are

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and the error approaches zero.

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Goal: divide a strip of paper into fifths.

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- Imperfect guessing is easy.
- Bisecting is easy.

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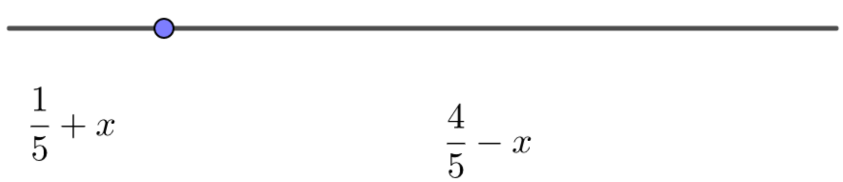
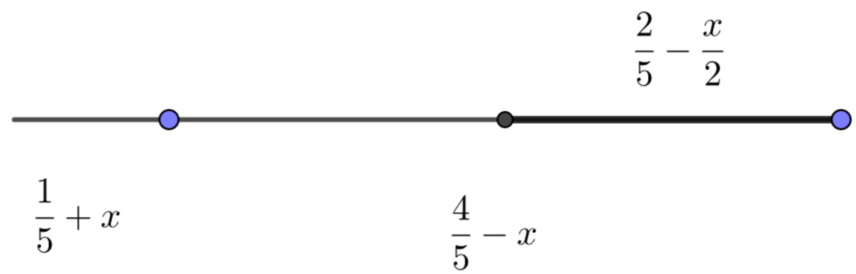


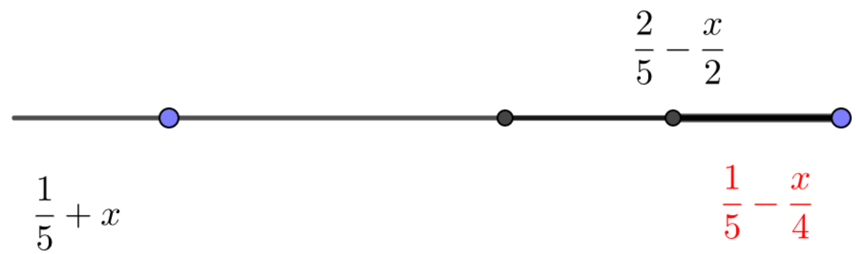
## Fujimoto folding

Next, bisect the remainder (on the right).



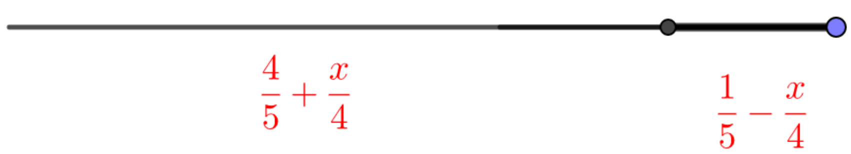
## Fujimoto folding

Bisect again.



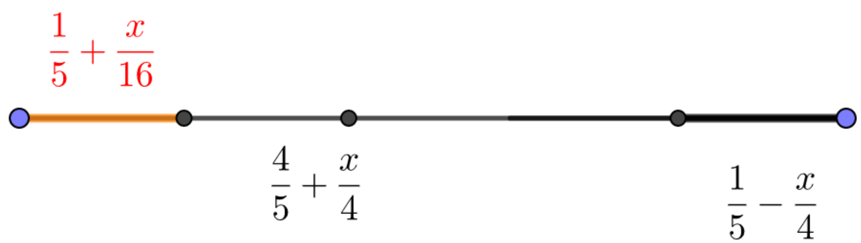
# Fujimoto folding

Notice that the error has been divided by 4.



## Fujimoto folding

Bisect the left twice, and the error is divided by 16!



# The arithmetic of folding

The sequence *RRLL*, repeated, *converges* to $1/5$. What if we wanted to get $1/7$?

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$$\frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \cdots = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}.$$

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So, *in binary* (base 2),

$$\frac{1}{7}=0.001001001001\ldots.$$

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$$0.00110011\ldots = \frac{3}{16} + \frac{3}{16^2} + \cdots = \frac{\frac{3}{16}}{1 - \frac{1}{16}} = \frac{\frac{3}{16}}{\frac{15}{16}} = \frac{3}{15} = \frac{1}{5}.$$

## The arithmetic of folding: in general

Think of the strip of paper as a length on the number line, starting at 0
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$$x + \frac{1}{2}(1 - x) = \frac{1}{2} + \frac{x}{2} = 0.1abcd\ldots.$$

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In other words, $L$ inserts a 0 into the binary representation of $x$, and $R$ inserts a 1. Any sequence of $L$s and $R$s will converge to whatever that binary number is (replacing $L$ with 0 and $R$ with 1).

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(in other words, *LRLRLR* ...)
and start with an absurd guess (close to 1), to end up with an excellent approximation to $\frac{1}{3}$.

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Tie a *neat* knot with a straw wrapper or ribbon. What do you get?

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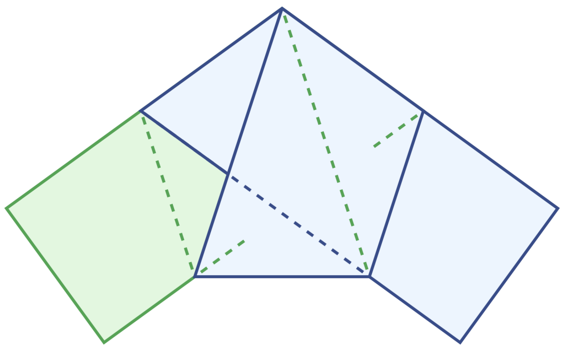
A regular PENTAGON!



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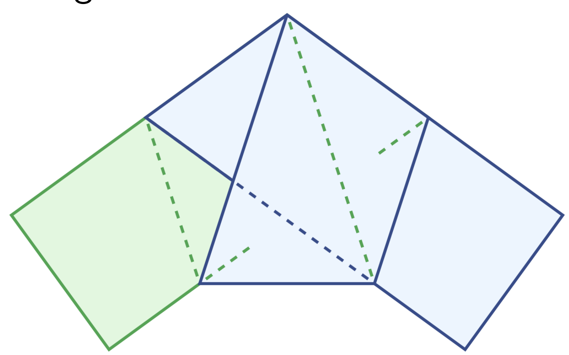
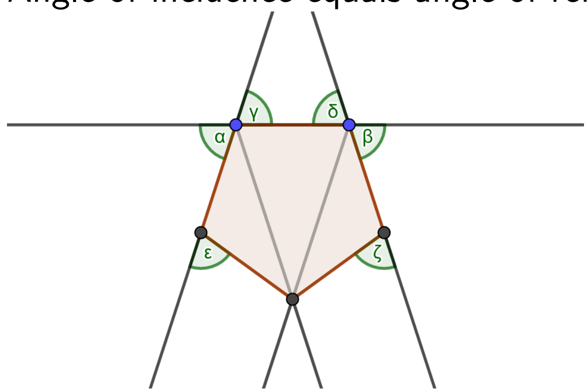


*Why* do you get it?

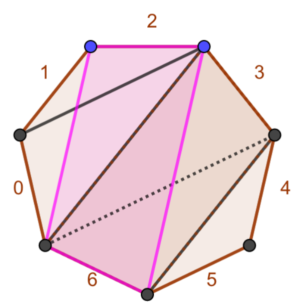
# Knots and numbers

Angle of incidence equals angle of reflection!



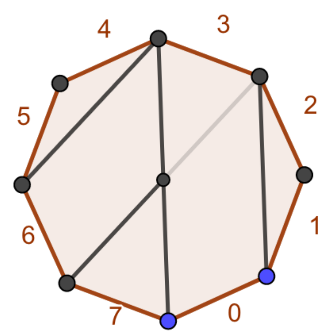
# Knots and numbers

Heptagon is theoretically possible!



# Knots and numbers

Even octagons are theoretically possible!



## Knots and numbers

But not squares or regular hexagons or equilateral triangles!

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Reason: Number theory!

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The Euler phi-function $\phi(n)$ is the number of positive integers less than or equal to $n$ that are relatively prime to $n$.
For example, $\phi(8) = 5$, since $1, 3, 5, 7$ are relatively prime to 8.
Modulo 8, the sequence

$$0, 3, 6, 9 \equiv 1, 4, 7, 10 \equiv 2, 5$$

covers all the residues modulo 8.

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If $n \neq 2, 3, 4, 6$, then $\phi(n) > 2$.

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So you can always find a "weaving interval"
But, for example, if $n = 6$, the only numbers relatively prime to 6 are 1 and 5, and these won't work for "weaving," since they send you to the adjacent side.